Integrable Lattice Geometry, Discriminant Separability and Pencils of Quadrics

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International Conference Geometric Methods in Mathematical Physics 2011
MSU, Moscow, 15 December 2011

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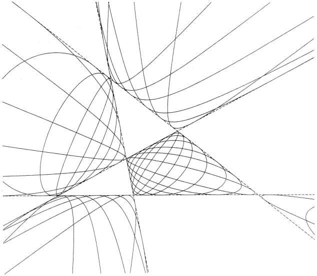
Pencil of conics



Two conics and tangential pencil

$$C_1 \ : \ a_0 w_1^2 + a_2 w_2^2 + a_4 w_3^2 + 2a_3 w_2 w_3 + 2a_5 w_1 w_3 + 2a_1 w_1 w_2 = 0$$
$$C_2 \ : \ w_2^2 - 4 w_1 w_3 = 0$$

$$F(s, x_1, x_2) = L(x_1, x_2)s^2 + K(x_1, x_2)s + H(x_1, x_2).$$

Theorem [V. D. (2009/2010)]

(i) There exists a polynomial $P = P(x)$ such that the discriminant of the polynomial F in s as a polynomial in variables x_1 and x_2 separates the variables:

$$\mathcal{D}_s(F)(x_1, x_2) = P(x_1)P(x_2). \tag{1}$$

(ii) There exists a polynomial $J = J(s)$ such that the discriminant of the polynomial F in x_2 as a polynomial in variables x_1 and s separates the variables:

$$\mathcal{D}_{x_2}(F)(s, x_1) = J(s)P(x_1). \tag{2}$$

Due to the symmetry between x_1 and x_2 the last statement remains valid after exchanging the places of x_1 and x_2.

Gauge equivalence

Gauge transformations

$$x \mapsto \frac{a_1 x + b_1}{c_1 x + d_1}$$
$$y \mapsto \frac{a_2 y + b_2}{c_2 y + d_2}$$
$$z \mapsto \frac{a_3 z + b_3}{c_3 z + d_3}$$

Discriminantly separable polynomials – definition [V. D. 2009/2010]

For a polynomial $F(x_1, \dots, x_n)$ we say that it is discriminantly separable if there exist polynomials $f_i(x_i)$ such that for every $i = 1, \dots, n$

$$\mathcal{D}_{x_i} F(x_1, \dots, \hat{x}_i, \dots, x_n) = \prod_{j \neq i} f_j(x_j).$$

It is symmetrically discriminantly separable if

$$f_2 = f_3 = \dots = f_n,$$

while it is strongly discriminatly separable if

$$f_1 = f_2 = f_3 = \dots = f_n.$$

It is weakly discriminantly separable if there exist polynomials $f_i^j(x_i)$ such that for every $i = 1, \dots, n$

$$\mathcal{D}_{x_i} F(x_1, \dots, \hat{x}_i, \dots, x_n) = \prod_{j \neq i} f_j^i(x_j).$$

Geometric interpretation of the Kowalevski fundamental equation

$$Q(w, x_1, x_2) := (x_1 - x_2)^2 w^2 - 2R(x_1, x_2)w - R_1(x_1, x_2) = 0$$

$$\begin{aligned}
R(x_1, x_2) =& -x_1^2 x_2^2 + 6\ell_1 x_1 x_2 + 2\ell c(x_1 + x_2) + c^2 - k^2 \\
R_1(x_1, x_2) =& -6\ell_1 x_1^2 x_2^2 - (c^2 - k^2)(x_1 + x_2)^2 - 4c\ell x_1 x_2 (x_1 + x_2) \\
& + 6\ell_1 (c^2 - k^2) - 4c^2 \ell^2
\end{aligned}$$

$$\begin{aligned}
a_0 =& -2 \qquad a_1 = 0 \qquad a_5 = 0 \\
a_2 =& 3\ell_1 \qquad a_3 = -2c\ell \qquad a_4 = 2(c^2 - k^2)
\end{aligned}$$

Geometric interpretation of the Kowalevski fundamental equation

Theorem [V. D. (2009)]

The Kowalevski fundamental equation represents a point pencil of conics given by their tangential equations

$$\hat{C}_1 : -2w_1^2 + 3l_1 w_2^2 + 2(c^2 - k^2)w_3^2 - 4clw_2 w_3 = 0;$$
$$C_2 : w_2^2 - 4w_1 w_3 = 0.$$

The Kowalevski variables w, x_1, x_2 in this geometric settings are the pencil parameter, and the Darboux coordinates with respect to the conic C_2 respectively.

Multi-valued Buchstaber-Novikov groups

n-valued group on X

$$m \ : \ X \times X \to (X)^n, \qquad m(x, y) = x * y = [z_1, \dots, z_n]$$

$(X)^n$ — symmetric n-th power of X

Associativity

Equality of two n^2-sets:

$$[x * (y * z)_1, \dots, x * (y * z)_n] \quad \text{и} \quad [(x * y)_1 * z, \dots, (x * y)_n * z]$$

for every triplet $(x, y, z) \in X^3$.

Unity e

$e * x = x * e = [x, \dots, x]$ for each $x \in X$.

Inverse $\mathrm{inv} : X \to X$

$e \in \mathrm{inv}(x) * x$, $e \in x * \mathrm{inv}(x)$ for each $x \in X$.

Two-valued group on $\mathbf{CP}^1$

The equation of a pencil

$F(s, x_1, x_2) = 0$

Isomorphic elliptic curves

$$\Gamma_1 \ : \ y^2 = P(x) \ \deg P = 4 \qquad \Gamma_2 \ : \ t^2 = J(s) \ \deg J = 3.$$

Two-valued group on $\mathbf{CP}^1$

There is a group structure on the cubic Γ_2. Together with its subgroup $\mathbf{Z}_2$, it defines the standard two-valued group structure on $\mathbf{CP}^1$:

$$s_1 *_c s_2 = \left[-s_1 - s_2 + \left(\frac{t_1 - t_2}{2(s_1 - s_2)} \right)^2, -s_1 - s_2 + \left(\frac{t_1 + t_2}{2(s_1 - s_2)} \right)^2 \right],$$

where $t_i = J'(s_i)$, $i = 1, 2$.

Theorem [V. D. (2009/2010)]

The general pencil equation after fractional-linear transformations

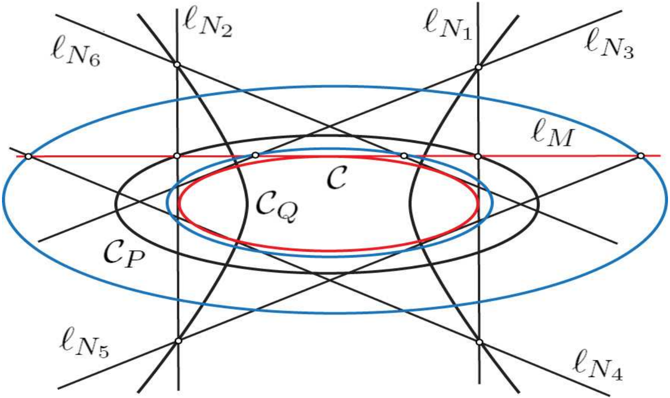
$$F(s, \hat{\psi}^{-1}(x_1), \hat{\psi}^{-1}(x_2)) = 0$$

defines the two valued group structure $(\Gamma_2, \mathbb{Z}_2)$ and the Kowalevski change of variables.

Two-valued group $\mathbf{CP}^1$

Theorem [V. D. (2009/2010)]

Associativity conditions for the group structure of the two-valued group $(\Gamma_2, \mathbb{Z}_2)$ and for its action on Γ_1 are equivalent to the great Poncelet theorem for a triangle.



Classification of the strongly discriminantly separable polynomials

Natural question: to classify discriminantly separable polynomials of degree two in each of three variables, up to gauge transformations.

Theorem (V. D. - K. Kukić, 2011)

All strongly discriminantly separable polynomials in three variables of degree two in each variable, with polynomial P with four simple roots, are gauge equivalent to the two valued group defined by the equation:

$$(x + y + z + \frac{g_2}{4}xyz)^2 - (4 + g_3xyz)(xy + yz + zx) = 0.$$

Classification of the strongly discriminantly separable polynomials

(B) (1,1,2): two simple zeros and one double zero, for canonical form $P(x) = x^2 - \epsilon^2$,

$$F_B = x_1x_2x_3 + \frac{\epsilon}{2}(x_1^2 + x_2^2 + x_3^2 - \epsilon^2),$$

(C) (2, 2): two double zeros, for canonical form $P(x) = x^2$,

$$F_{C1} = \alpha_1 x_1^2 x_3^2 + \alpha_2 x_1x_2x_3 + \alpha_3 x_2^2, \quad \alpha_2^2 - 4\alpha_1\alpha_3 = 1,$$

$$F_{C2} = \beta_1 x_1^2 x_2^2 x_3^2 + \beta_2 x_1x_2x_3 + \beta_3, \quad \beta_2^2 - 4\beta_1\beta_3 = 1,$$

(D) (1,3): one simple and one triple zero, for canonical form $P(x) = x$,

$$F_D = -\frac{1}{2}(x_1x_2 + x_2x_3 + x_1x_3) + \frac{1}{4}(x_1^2 + x_2^2 + x_3^2),$$

Classification-continuation

(E) (4): one quadruple zero, for canonical form $P(x) = 1$,

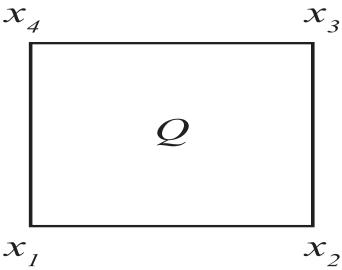
$$F_{E1} = \gamma_1(x_1+x_2+x_3)^2+\gamma_2(x_1+x_2+x_3)+\gamma_3, \quad \gamma_2^2-4\gamma_1\gamma_3 = 1,$$

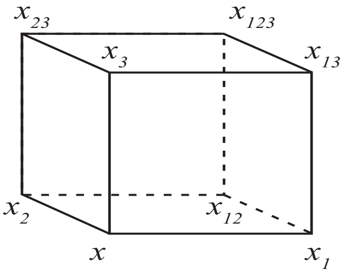
$$F_{E2} = \gamma_1(x_2+x_3-x_1)^2+\gamma_2(x_2+x_3-x_1)+\gamma_3, \quad \gamma_2^2-4\gamma_1\gamma_3 = 1,$$

$$F_{E3} = \gamma_1(x_1+x_3-x_2)^2+\gamma_2(x_1+x_3-x_2)+\gamma_3, \quad \gamma_2^2-4\gamma_1\gamma_3 = 1,$$

$$F_{E4} = \gamma_1(x_1+x_2-x_3)^2+\gamma_2(x_1+x_2-x_3)+\gamma_3, \quad \gamma_2^2-4\gamma_1\gamma_3 = 1.$$

Integrable quad-graphs





Toward Adler-Bobenko-Suris quad graphs: h

From F to $\hat{h}$

$$\hat{h}(x_1, x_2, \alpha) = \frac{F(x_1, x_2, \alpha)}{\sqrt{P(\alpha)}}$$

The system for h_B

$$h_{22} = 0,\ h_{21} = h_{12} = 0,\ h_{01} = h_{10} = 0$$

$$h_{02} = h_{20},\ h_{11} = \pm\sqrt{1 + 4b_{20}^2},\ h_{00} = \frac{e^2}{4b_{20}}.$$

h_{20} arbitrary function of α. ABS2009: $h_{20} = \alpha/(1 - \alpha^2)$.

$\hat{h}_B$ and $\hat{Q}_B$

$\hat{h}_B$

$$\hat{h}_{20} = e/2\sqrt{\alpha^2 - e^2}$$

$$\hat{h}_B(x_1, x_2, \alpha) = \left(\frac{e}{2}(x_1^2 + x_2^2 + \alpha^2) + \alpha x_1 x_2 - \frac{e^3}{2}\right) / \sqrt{\alpha^2 - e^2}$$
$$= F_B(x_1, x_2, \alpha)/\sqrt{\alpha^2 - e^2}.$$

$\hat{Q}_B = \sqrt{\beta_1^2 - e^2}(x_1x_4 + x_2x_3) + \sqrt{\alpha_1^2 - e^2}(x_1x_2 + x_3x_4) +$

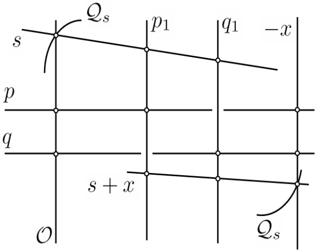
$$\frac{\alpha_1\sqrt{\beta_1^2 - e^2} + \beta_1\sqrt{\alpha_1^2 - e^2}}{e}(x_1x_3 + x_2x_4)$$
$$- \frac{\sqrt{\beta_1^2 - e^2}\sqrt{\alpha_1^2 - e^2}(\alpha_1\sqrt{\beta_1^2 - e^2} + \beta_1\sqrt{\alpha_1^2 - e^2})}{e}.$$

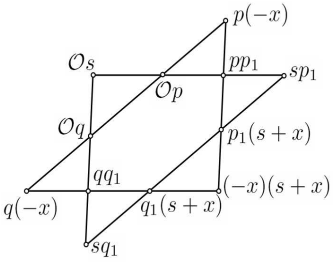
Our book: V. D, M. Radnovic, Poncelet Porisms and Beyond, Springer 2011, Russian version RCD 2010

Motto of the book [Griffiths-Harris 1978]

"One of the most important and also most beautiful theorems in classical geometry is that of Poncelet (...) His proof was synthetic and somewhat elaborate in what was to become the predominant style in projective geometry of last century. Slightly thereafter, Jacobi gave another argument based on the addition theorem for elliptic functions. In fact, as will be seen below, the Poncelet theorem and addition theorem are essentially equivalent, so that at least in principle Poncelet gave a synthetic derivation of the group law on an elliptic curve. Because of the appeal of the Poncelet theorem it seems reasonable to look for higher-dimensional analogues... Although this has not yet turned out to be the case in the Poncelet-type problems..."

Six-pointed star theorem and quad graphs





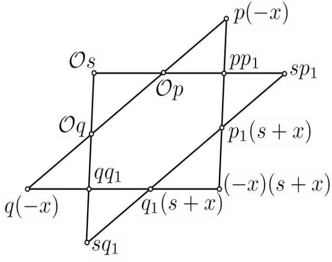
Six-pointed star theorem, [V.D, M. R. 2008]

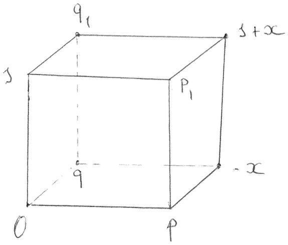
Let $\mathcal{F}$ be a family of confocal quadrics in $\mathbf{P}^3$. There exist configurations consisting of 12 planes in $\mathbf{P}^3$ with the following properties:

- The planes may be organized in 8 triplets, such that each plane in a triplet is tangent to a different quadric from $\mathcal{F}$ and the three touching points are collinear. Every plane in the configuration is a member of two triplets.
- The planes may be organized in 6 quadruplets, such that the planes in each quadruplet belong to a pencil and they are tangent to two different quadrics from $\mathcal{F}$. Every plane in the configuration is a member of two quadruplets.

Moreover, such a configuration is determined by three planes tangent to three different quadrics from $\mathcal{F}$, with collinear touching points.

Operational consistency: quad graph interpretation [V. D., M. Radnovic 2011]





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