Flat 3-webs via semi-simple Frobenius 3-manifolds

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Geometrical Methods in Mathematical Physics
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Frobenius manifold
geometric definition

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- Local flow of E re-scales Multiplication and Metric,
- 4-tensor $(\nabla_z c)(u, v, w)$ is symmetric in u, v, w, z, where $c(u, v, w) :=< u \cdot v, w >$.

3-web via Frobenius 3-manifold

Let the algebra T_pM be semi-simple:

$$T_pM = \mathbb{C}\{e_1\} \otimes \mathbb{C}\{e_2\} \otimes \mathbb{C}\{e_3\}$$

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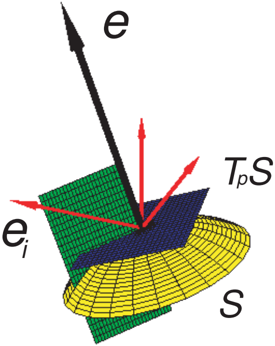

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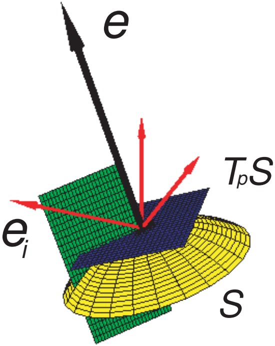

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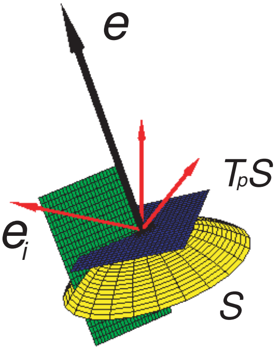

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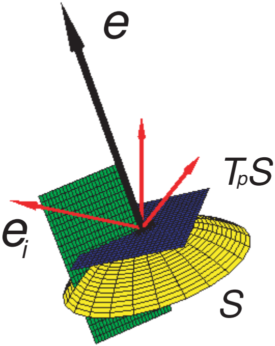

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The constructed 3-web will be called booklet 3-web.

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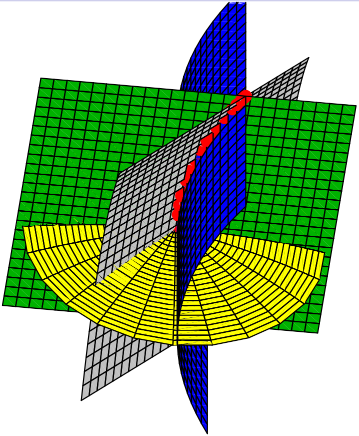

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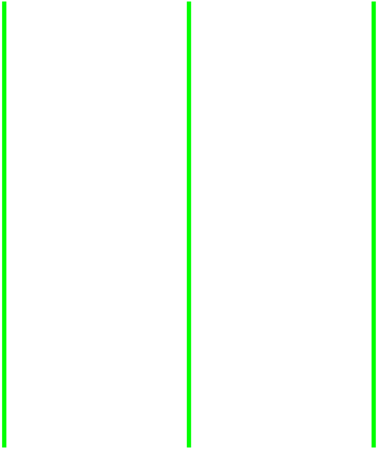
holonomy of 3-webs

$$\mathcal{W} = \{\mathcal{F}_1,$$



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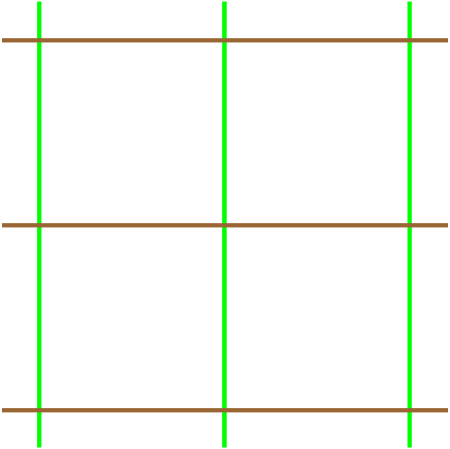
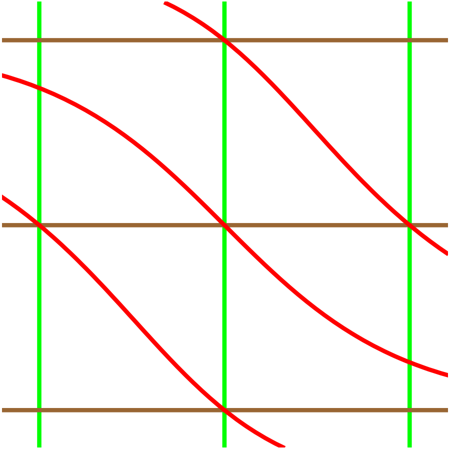
$$\mathcal{W} = \{\mathcal{F}_1, \mathcal{F}_2,$$



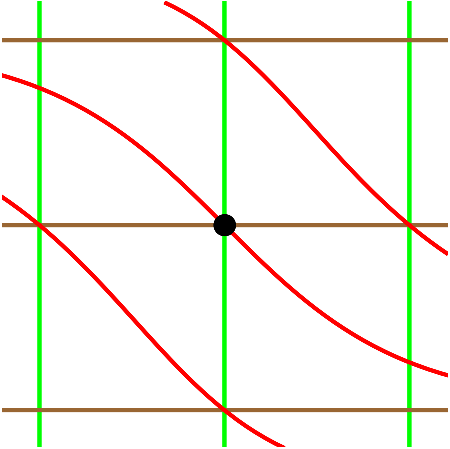
holonomy of 3-webs

$$\mathcal{W} = \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\} = \mathcal{W}(x, y, x + y + \frac{1}{5}xy(x - y))$$



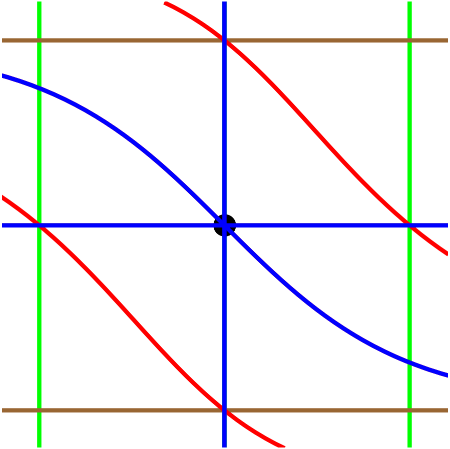
holonomy of 3-webs

$0 \in \mathbb{C}^2$



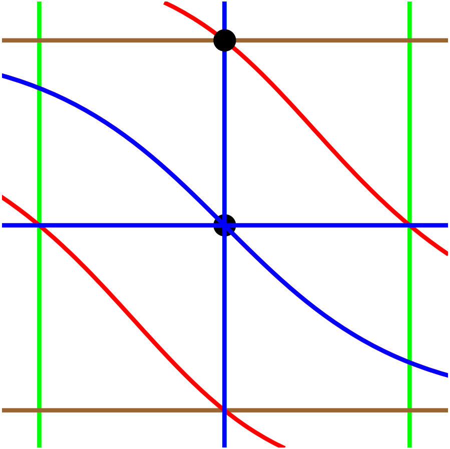
holonomy of 3-webs

$$0 \in L_1 \cap L_2 \cap L_3$$



holonomy of 3-webs

$p_1 \in L_1$



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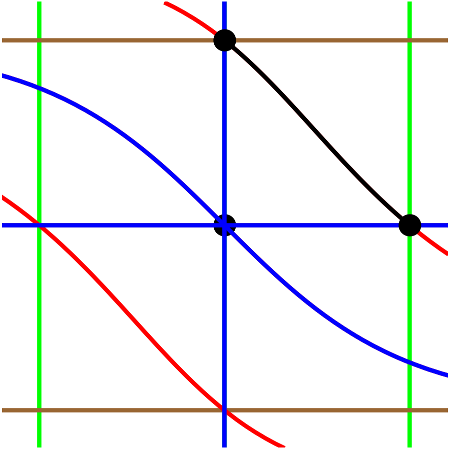
$$h_{12}(p_1) = p_2 \in L_2, \quad h_{12} : (L_1, 0) \to (L_2, 0)$$



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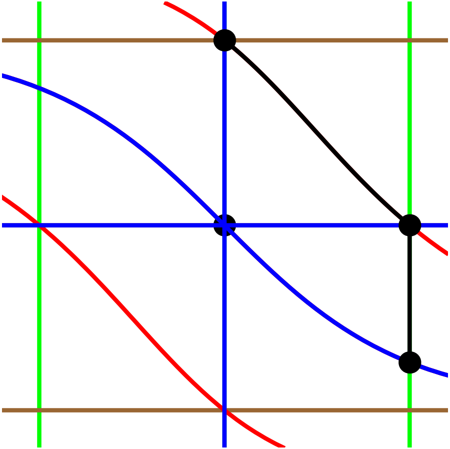

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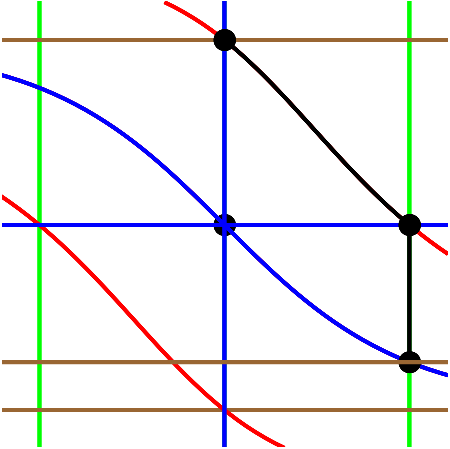

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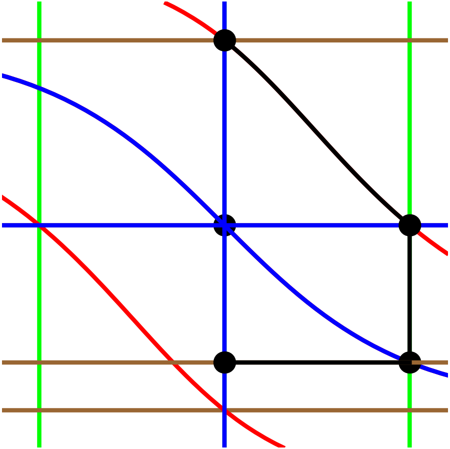
$$h_{31}(p_3) = p_4 \in L_1, \quad h_{31} : (L_3, 0) \to (L_1, 0)$$



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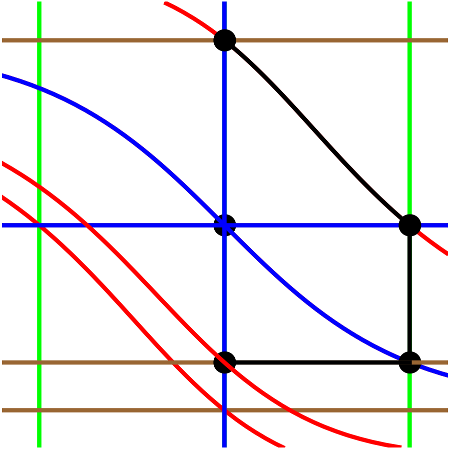

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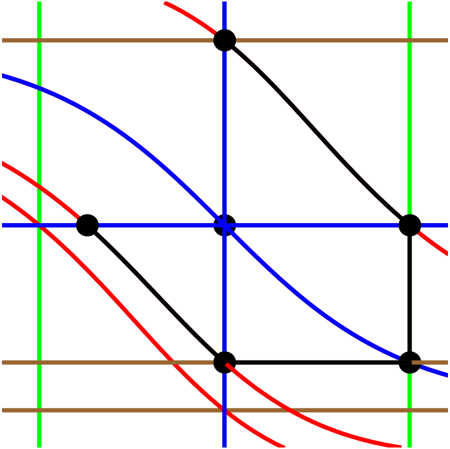

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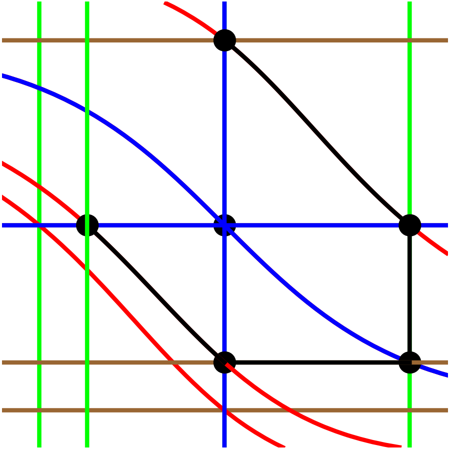

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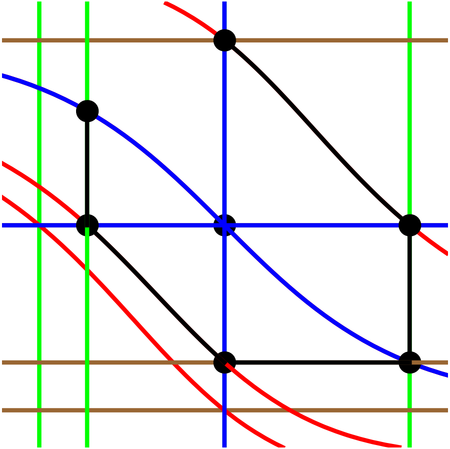

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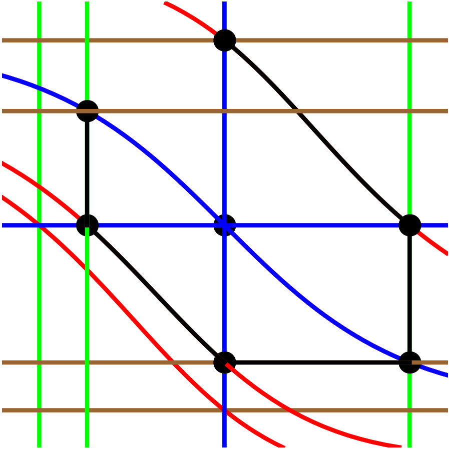

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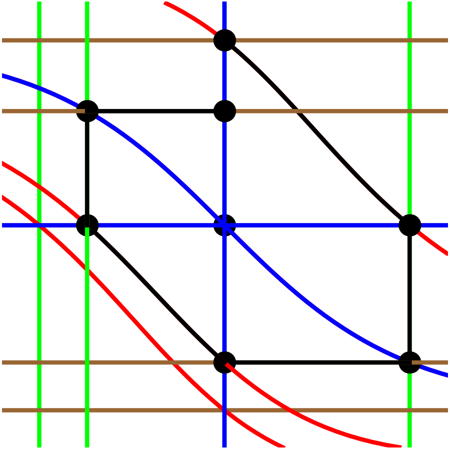
$$h_{31}(p_6) = (h_{31} \circ h_{23} \circ h_{12} \circ h_{31} \circ h_{23} \circ h_{12})(p_1) = p_7 \in L_1$$



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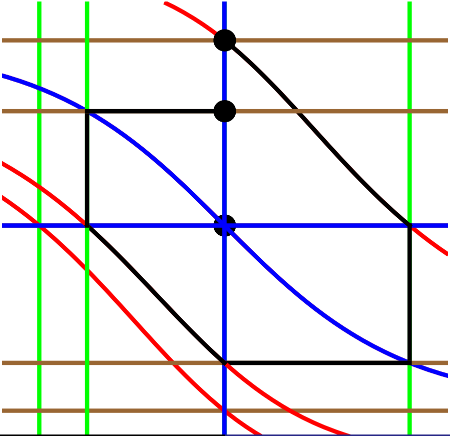
defines holonomy of the 3-web at 0



hexagonal 3-web

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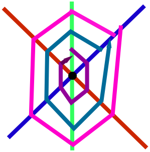

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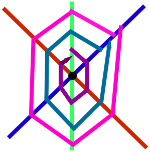

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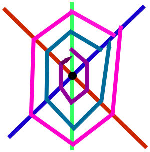

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3-web is hexagonal if some local diffeomorphism maps its foliations in 3 families of parallel lines.

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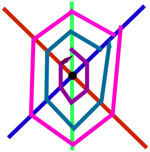

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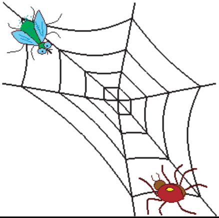
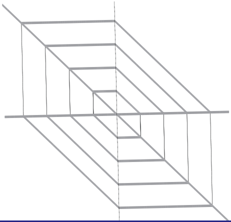
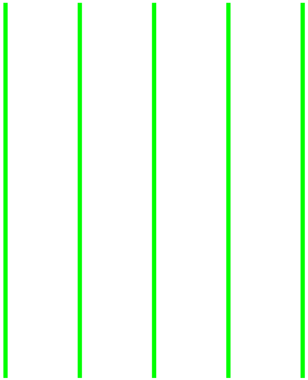


$$\xrightarrow{\varphi}$$



regular hexagonal 3-web

$$\mathcal{W}(x, y, x+y)$$



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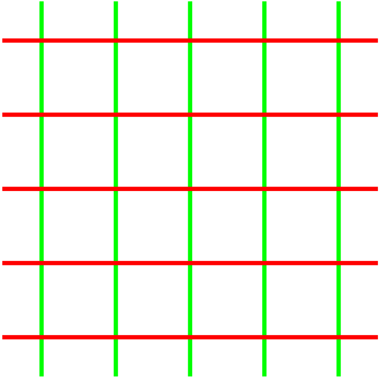

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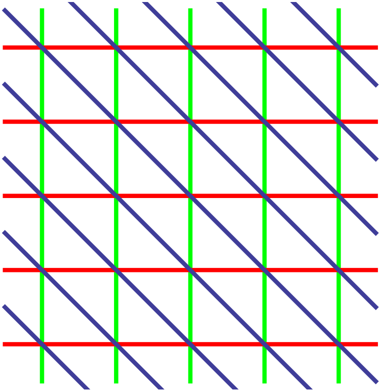

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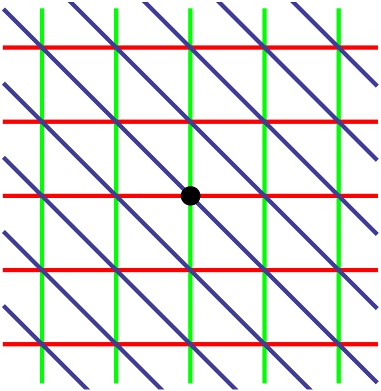

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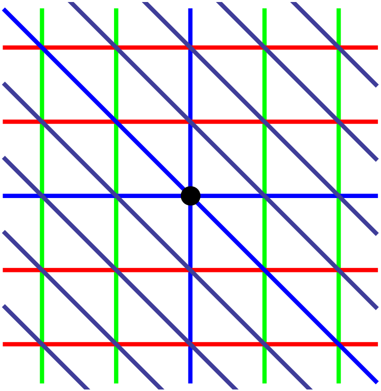

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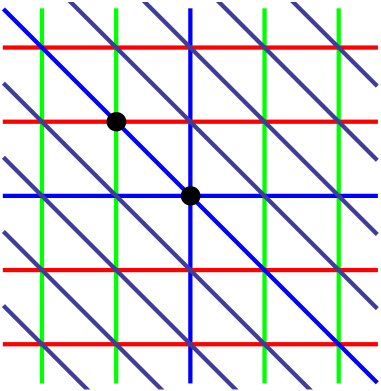

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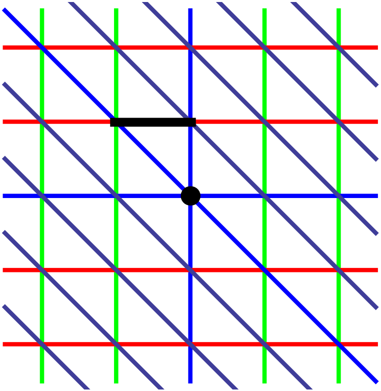

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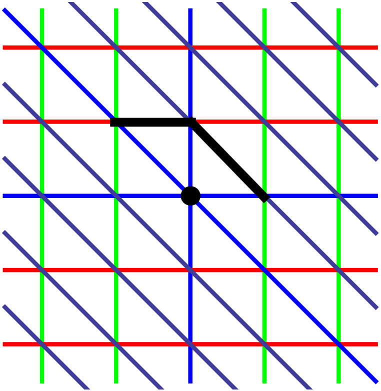

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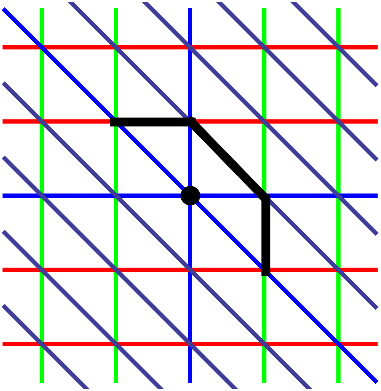

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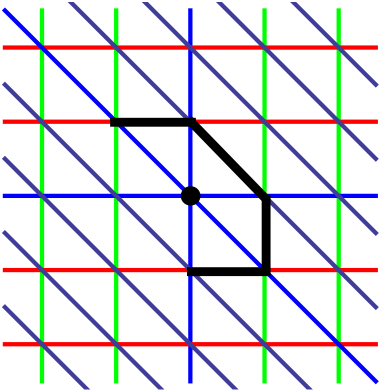

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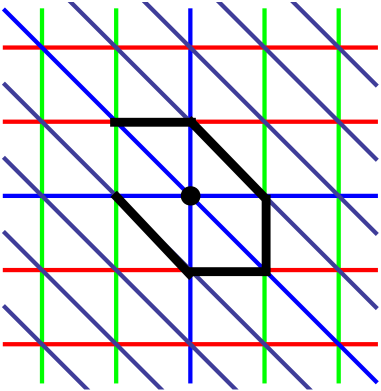

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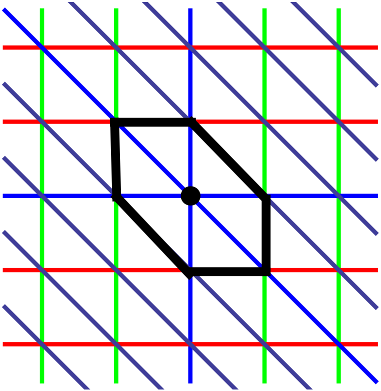

booklet 3-web is hexagonal

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Theorem (Dubrovin coordinates)

M is massive $\Rightarrow$ $\exists$ coordinates $\lambda^1, ..., \lambda^n$:
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The web is defined by $\omega_i := d\theta_i$,

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1) idempotents are partial differentiations: $e_i := \partial_{\lambda^i}$.
2)$E = \sum \lambda^i \partial_{\lambda^i}$ holds true.
To find them, solve $E \cdot \xi = \lambda \xi$.

Corollary

Booklet 3-web is flat.

Let $\imath : U \in \mathbb{C}^2 \to S \subset M$ be a local parametrization.
The web is defined by $\omega_i := d\theta_i$, where
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characteristic webs

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associativity equation

Frobenius structure on 3-manifolds is defined by weighted homogeneous solution of WDVV equation.

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Theorem

Booklet 3-web is bi-holomorphic to the characteristic 3-web.

singularities of flat 3-webs

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regular and singular points

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- Point (x, y) is regular $\Leftrightarrow$ all directions are distinct.

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“Singularity is almost invariably a clue.“

A. Conan Doyle

examples of good singularities

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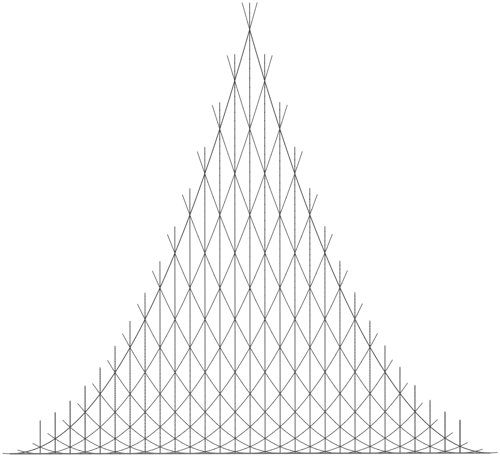

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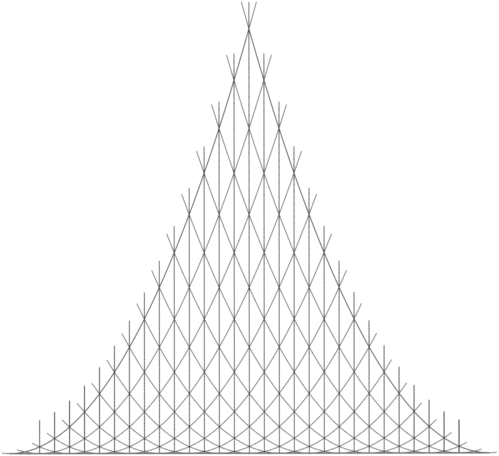

leaves are tangent to the discriminant curve

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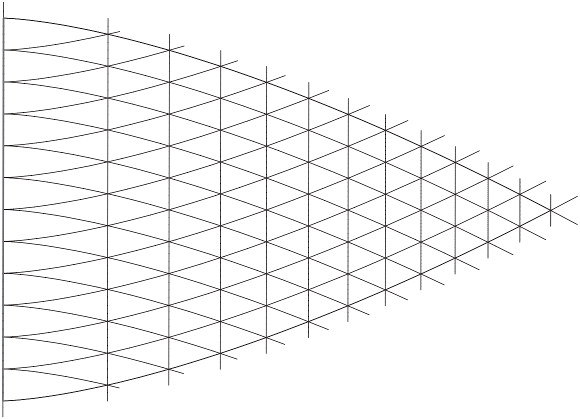

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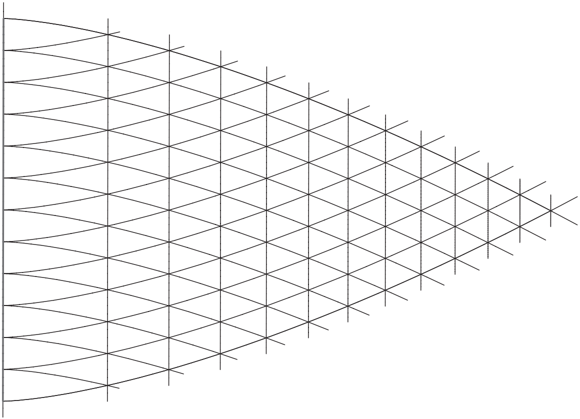

leaves have cusps on the discriminant curve.

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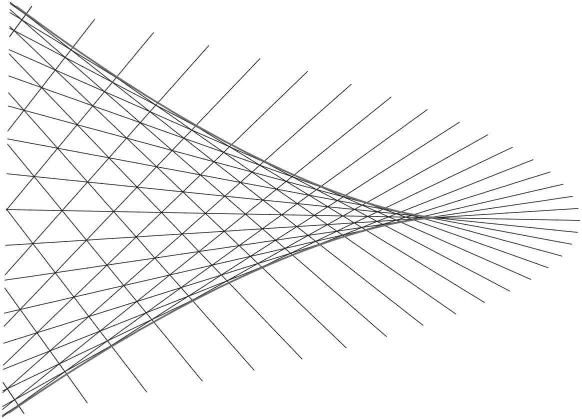

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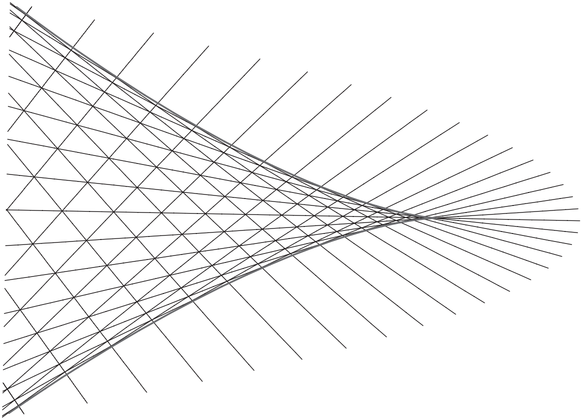

Clairaut Equation $p^3 + px - y = 0.$

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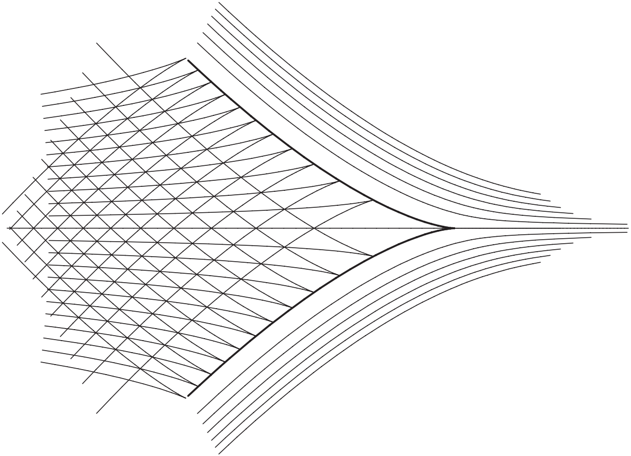

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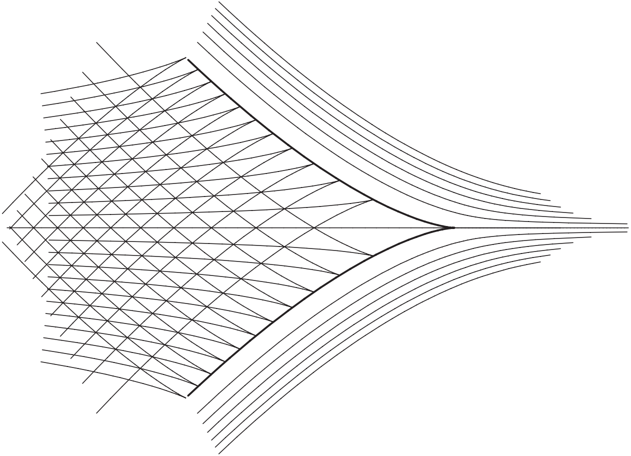
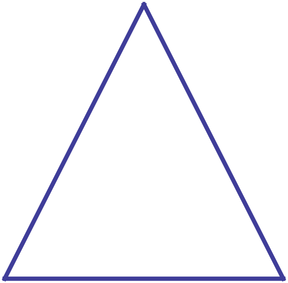


Equation $p^3 + 2px + y = 0.$

examples of bad singularities



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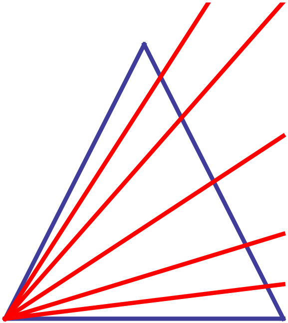

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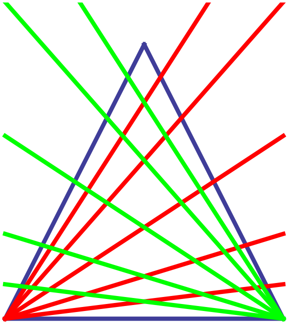

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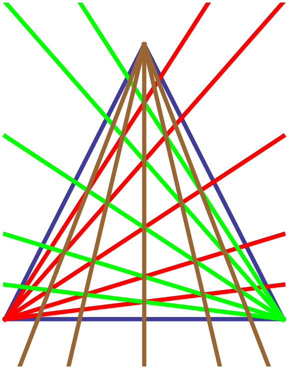

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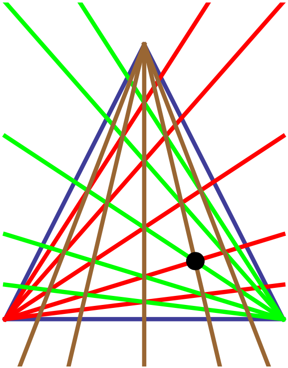

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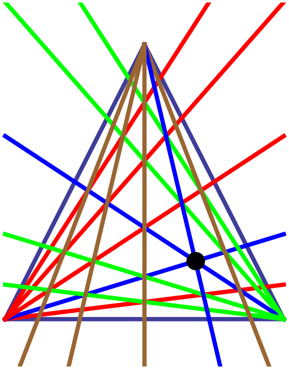

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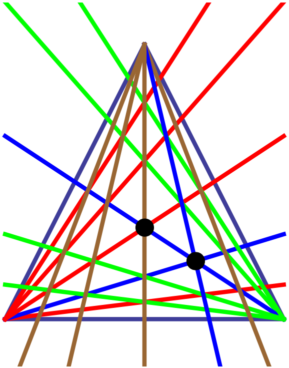

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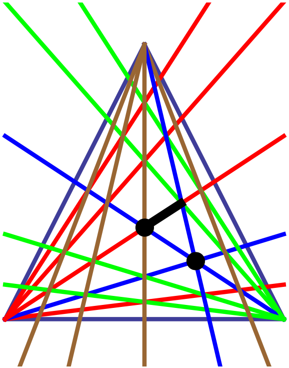

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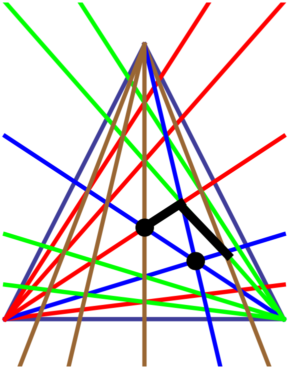

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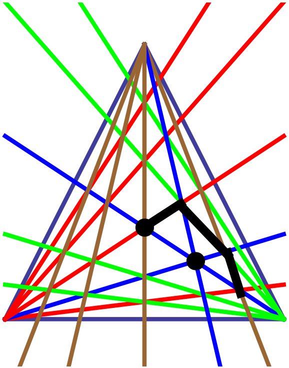

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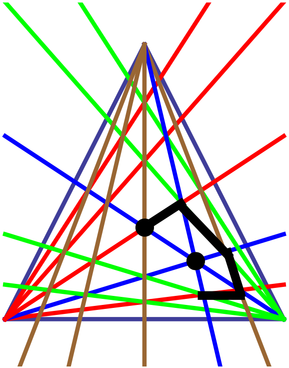

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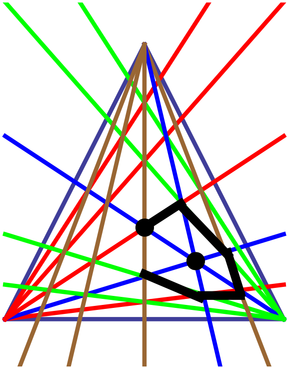

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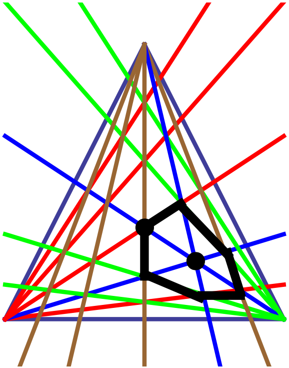

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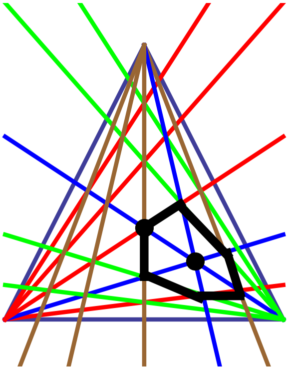
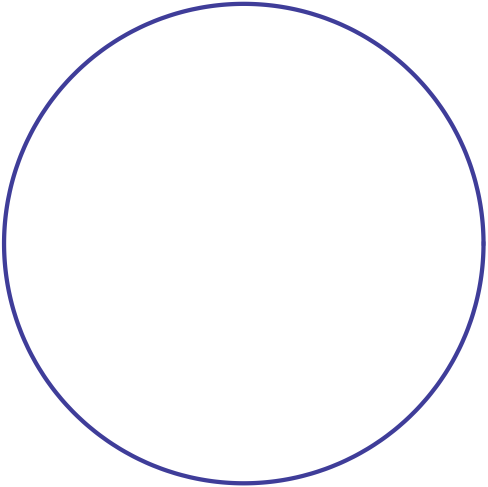


Direction field is not defined at the vertices!

examples of bad singularities



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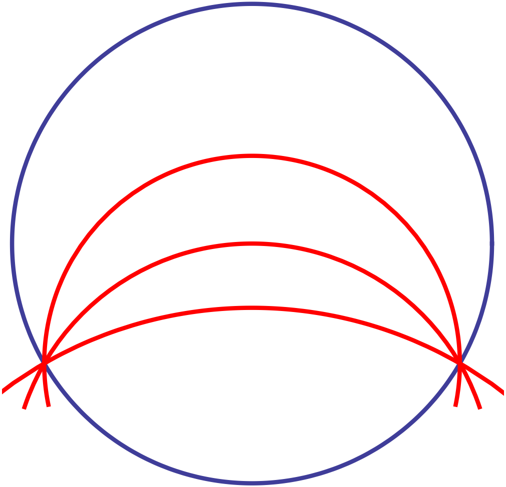

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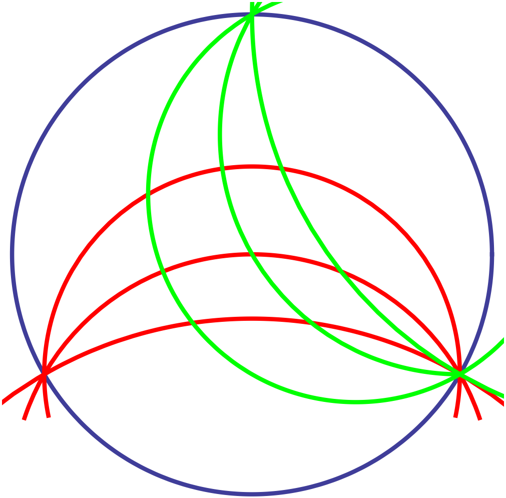

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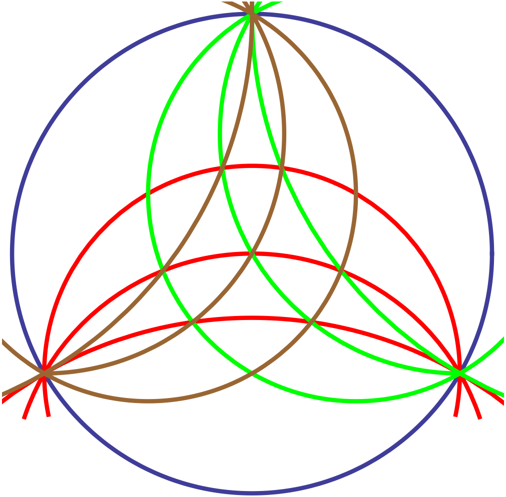

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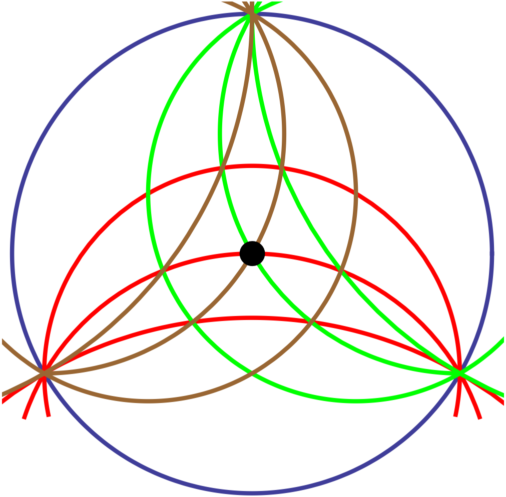

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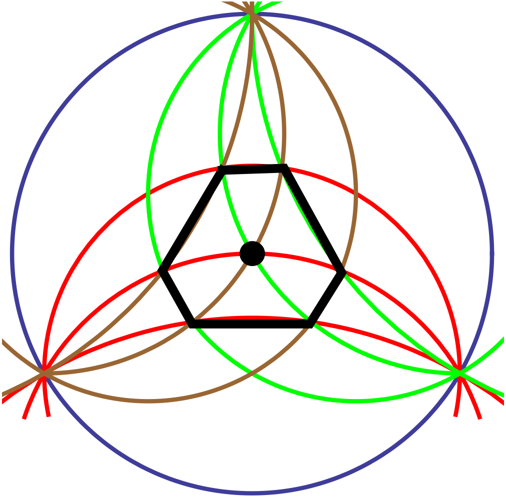

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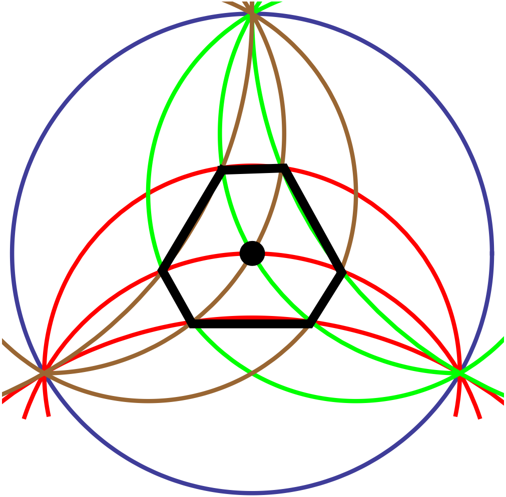

focus singularities

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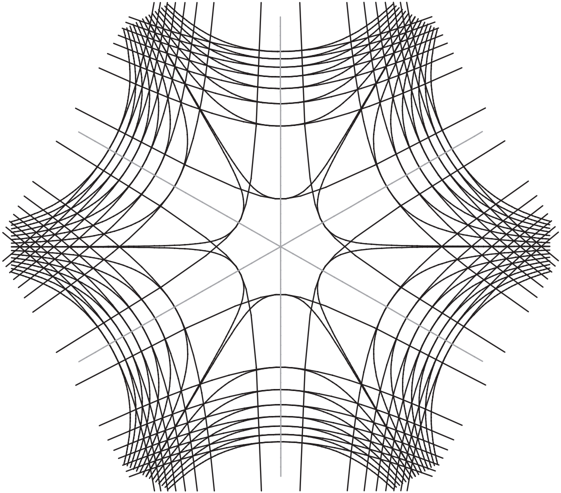

saddle singularity

infinitesimal symmetries

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Definition

Infinitesimal symmetry is a vector field $X = \xi(x,y)\partial_x + \eta(x,y)\partial_y$, whose local flow preserves the web.

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examples

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At a regular point (x,y) a 3-web:

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infinitesimal symmetries

Definition

Infinitesimal symmetry is a vector field $X = \xi(x, y)\partial_x + \eta(x, y)\partial_y$, whose local flow preserves the web.

examples

- $dy = 0$ has a symmetry algebra $\{X = \xi(x, y)\partial_x + \eta(y)\partial_y\}$.
- $dx \cdot dy = 0$ has a symmetry algebra $\{X = \xi(x)\partial_x + \eta(y)\partial_y\}$.

Theorem [Cartan]

At a regular point (x, y) a 3-web:

- either does not have symmetries (generic case),
- or has an 1-dimensional symmetry algebra and is equivalent to $dx \cdot dy \cdot (dy + u(x + y)dx) = 0$ with the symmetry $\partial_y - \partial_x$,
- or has a 3-dimensional symmetry algebra and is equivalent to $dx \cdot dy \cdot (dy + dx) = 0$ with the algebra $\{\partial_x,\ \partial_y, x\partial_x + y\partial_y\}$.

symmetries at singular points

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Let $p \in S$ and C_p be the orbit of p under the flow $\exp(s \cdot e)$.
$T_a(p) := \exp(a \cdot E) C_p \cap S$ is a symmetry.

web curvature and Chern connection

web curvature and Chern connection

cubic implicit ODEs

3-web can be described by $p^3 + A(x,y)p + B(x,y) = 0$.

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differential forms

form σ_i vanishes on the i-th family of web leaves

$\sigma_1{=}dy{-}p_1\,dx,\quad \sigma_2{=}dy{-}p_2\,dx,\quad \sigma_3{=}dy{-}p_3\,dx$

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normalization: $\sigma_1 + \sigma_2 + \sigma_3 = 0$

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$\gamma := h_2\sigma_1 - h_1\sigma_2 = \ldots$, where $d\sigma_i = h_i\Omega$

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$\Rightarrow \exists\, u_i :\ d(u_i) = f\sigma_i \Rightarrow u_1 + u_2 + u_3 = 0$ (hexagonality)

geometry of Chern connection

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connection on TM

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- Suppose $v = \eta^1 e_1 + \eta^2 e_2 + \eta^3 e_3 \in T_pS$ and a curve $\alpha : I \mapsto S$ $\alpha(0) = p$ does not passes through singular points of the 3-web.

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- The connection form of ODE $p^3+A(x,y)p+B(x,y)=0$
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Booklet 3-web has holomorphic Chern connection.

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Booklet 3-web has holomorphic Chern connection.

For the 1st canonical form of WDVV, $\gamma = 0$ holds true in adjusted flat coordinates of M, restricted on S.

normal forms for possible singularities

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Cubic ODE with closed holomorphic Chern connection and an infinitesimal symmetry is locally equivalent to one of the list:

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Cubic ODE with closed holomorphic Chern connection and an infinitesimal symmetry is locally equivalent to one of the list:

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4. $p^3 + 4x(y - \frac{4}{9}x^3)p + y^2 + \frac{64}{81}x^6 - \frac{32}{9}yx^3 = 0$,
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6. $p^3 + y^{3+m_0}p = y^{\frac{9+3m_0}{2}}F\left([(m_0+1)]\, xy^{\frac{1+m_0}{2}}\right)$
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where m_0 *is non-negative integer and* $F(t)$ *solves*

$[12 + 2t^2 - 9tF]\frac{dF}{dt} = \frac{2(m_0+3)}{m_0+1}(4 + 27F^2)$ *with* $F(0) = 0$.

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where m_0 *is non-negative integer and* $F(t)$ *solves* $[12 + 2t^2 - 9tF]\frac{dF}{dt} = \frac{2(m_0+3)}{m_0+1}(4+27F^2)$ *with* $F(0) = 0$.
The weights $[w_1 : w_2]$ *determine uniquely the normal form.*

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The construction is easily generalized to higher dimensions.

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The construction is easily generalized to higher dimensions.
For n-dimensional Frobenius manifold, we get
n commuting vector fields v_i in $\mathbb{C}^{n-1}$, satisfying $\sum_{i=1}^{n} v_i = 0$;

problem

Suppose the web has a symmetry and its Chern connection form is closed and holomorphic.

- Does this web define Frobenius structure?
- If it does, is this structure unique?
- If not, what are the moduli?

observation

The construction is easily generalized to higher dimensions.
For n-dimensional Frobenius manifold, we get
n commuting vector fields v_i in $\mathbb{C}^{n-1}$, satisfying $\sum_{i=1}^{n} v_i = 0$;
a flat n-web of curves in $\mathbb{C}^{n-1}$, admitting a"linear" symmetry.

on a general classification of cubic ODEs

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Curvature K of the solution 3-web is an obstacle for a general classification of implicit cubic ODEs.

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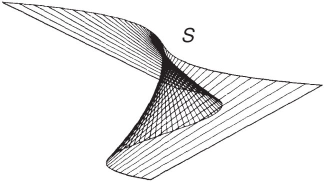
- Implicit ODE $F(x, y, p) = 0$ defines a surface in $\mathbb{C}^2 \times \mathbb{P}^1(\mathbb{C})$: $S := \{(x, y, p) : F(x, y, p) = 0\}$.



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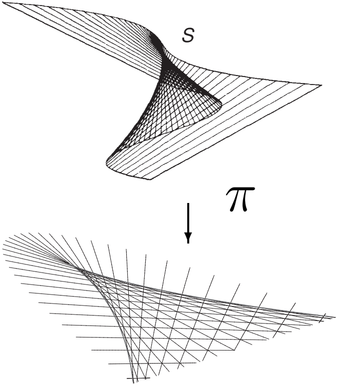

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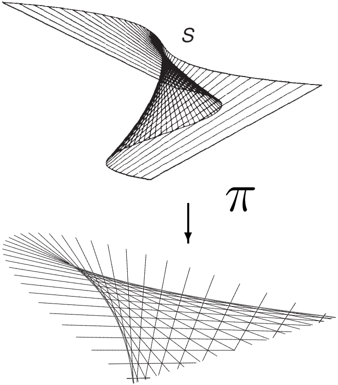

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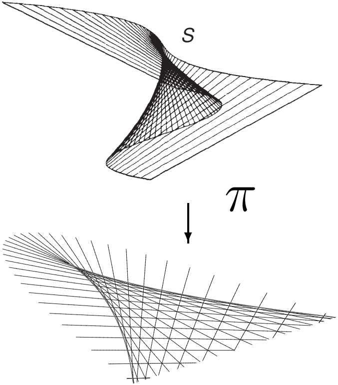

partial classification result

S and C are smooth

Theorem (AS'08)

If an implicit cubic ODE $p^3 + a(x, y)p^2 + b(x, y)p + c(x, y) = 0$ *has a flat web of solutions and satisfy regularity condition at* $m = (x_0, y_0, p_0) \in C \subset S$ *then there is a local diffeomorphism at* $\pi(m) = (x_0, y_0)$*, reducing this ODE to:*

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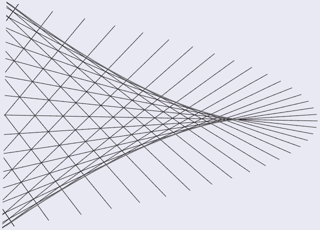

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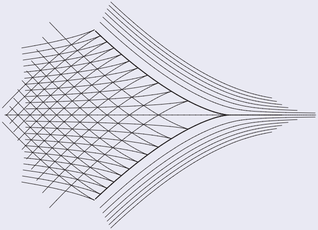

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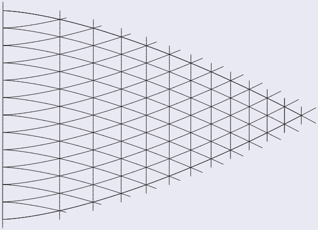

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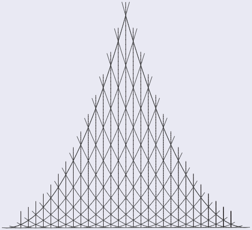

web theory founders

Blaschke, Wilhelm 1885-1962

Chern, S.-S. 1911-2004