Deformations of exact and homogeneous Poisson pencils of hydrodynamic type

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Based on joint works with A. Arsie and with G. Falqui

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Plan of the talk

- An example: the scalar case.
- Deformations of semisimple Poisson pencils of hydrodynamic type and their central invariants. (Dubrovin, Liu and Zhang).
- Deformations of exact, semisimple Poisson pencils of hydrodynamic type (joint work with Gregorio Falqui: to appear in Physica D).
- The case of non constant central invariants (joint work with A. Arsie: **arXiv:1107.2327**).

The scalar case

$$\omega_\lambda = \omega_2 - \lambda\omega_1 = u\delta'(x-y) + \frac{1}{2}u_x\delta(x-y) - \lambda\delta'(x-y)$$

It is exact:

$$\begin{aligned} \mathrm{Lie}_e\omega_1 &= 0 \\ \mathrm{Lie}_e\omega_2 &= \omega_1. \end{aligned}$$

with $e = \frac{\partial}{\partial u}$.
It is homogeneous:

$$\begin{aligned} \mathrm{Lie}_E\omega_1 &= (d-2)\omega_1 \\ \mathrm{Lie}_E\omega_2 &= (d-1)\omega_2. \end{aligned}$$

with $d = 0$ and $E = u\frac{\partial}{\partial u}$.

$$\Pi^{ij}_{\lambda} = \omega_{\lambda} + \sum_{k=1}^{\infty} \epsilon^{2k} P_2^{(k)}.$$

Second order deformation (PL, 2002):

$$P_2^{(2)} = \left\{ \partial_x^2 \left(c_2 \delta^{(1)}(x-y) \right) + c_2 \delta^{(3)}(x-y) + (\partial_x c_2) \delta^{(2)}(x-y) \right\}$$

where $c_2 = c_2(u)$ is an arbitrary function.

Fourth order:

$$P_2^{(4)} = \left\{ \partial_x^4 \left(c_4 \delta^{(1)}(x-y) \right) + c_4 \delta^{(5)}(x-y) + (\partial_x c_4) \delta^{(4)}(x-y) \right\}$$

where $c_4 = \frac{\partial}{\partial u}(c_2)^2$.

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Sixth order (A. Arsie, PL, 2011):

$$
\begin{aligned}
&P_2^{(6)} = \\
&= \left\{\partial_x^6\left(c_6\delta^{(1)}(x-y)\right) + c_6\delta^{(7)}(x-y) + (\partial_x c_6)\delta^{(6)}(x-y)\right\} \\
&+ \left\{h\delta^{(3)}(x-y) + (\partial_x h)\delta^{(2)}(x-y) + \partial_x^2\left(h\delta^{(1)}(x-y)\right)\right\} \\
&+ \left\{\partial_x\left((\partial_x^3 g)\delta^{(3)}(x-y)\right) + (\partial_x^2 g)\delta^{(5)}(x-y) + (\partial_x^3 g)\delta^{(4)}(x-y)\right. \\
&\left.+\partial_x^3\left((\partial_x^2 g)\delta^{(2)}(x-y)\right)\right\},
\end{aligned}
$$

where

$$
c_6 = -\frac{1}{2}\frac{\partial}{\partial u}\left(c_2^2\,\frac{\partial c_2}{\partial u}\right),
$$

$$
g = \frac{1}{2}\int\left\{\frac{3}{2}c_2^2\,\frac{\partial^3 c_2}{\partial u^3} + \left(\frac{\partial c_2}{\partial u}\right)^3 + \frac{19}{3}c_2\,\frac{\partial^2 c_2}{\partial u^2}\,\frac{\partial c_2}{\partial u}\right\}du
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$$

$$
\begin{aligned}
h &= \left[\frac{97}{60}c_2\left(\frac{\partial^2 c_2}{\partial u^2}\right)^2 + \frac{8}{3}\left(\frac{\partial c_2}{\partial u}\right)^2\frac{\partial^2 c_2}{\partial u^2} + \frac{21}{40}c_2^2\frac{\partial^4 c_2}{\partial u^4}\right. \\
&\left. + \frac{49}{15}c_2\left(\frac{\partial^3 c_2}{\partial u^3}\right)\frac{\partial c_2}{\partial u}\right]u_{xx}^2 + \left[\frac{254}{3}\left(\frac{\partial c_2}{\partial u}\right)^2\frac{\partial^4 c_2}{\partial u^4} + \frac{17}{5}c_2^2\frac{\partial^6 c_2}{\partial u^6}\right. \\
&+ \frac{176}{3}c_2\left(\frac{\partial^3 c_2}{\partial u^3}\right)^2 + \frac{4018}{45}c_2\left(\frac{\partial^4 c_2}{\partial u^4}\right)\frac{\partial^2 c_2}{\partial u^2} + \frac{1684}{45}c_2\frac{\partial^5 c_2}{\partial u^5}\frac{\partial c_2}{\partial u} \\
&\left. + \frac{14512}{45}\left(\frac{\partial c_2}{\partial u}\right)\left(\frac{\partial^2 c_2}{\partial u^2}\right)\frac{\partial^3 c_2}{\partial u^3}\right]u_x^4 + \left[\frac{3}{10}c_2^2\frac{\partial^4 c_2}{\partial u^4} + \right. \\
&\left. \frac{2}{3}\left(\frac{\partial c_2}{\partial u}\right)^2\frac{\partial^2 c_2}{\partial u^2} + \frac{1}{15}c_2\left(\frac{\partial^2 c_2}{\partial u^2}\right)^2 + \frac{28}{15}c_2\left(\frac{\partial^3 c_2}{\partial u^3}\right)\frac{\partial c_2}{\partial u}\right]u_{xxx}u_x \\
&+ \left[\frac{139}{10}\left(\frac{\partial c_2}{\partial u}\right)\left(\frac{\partial^2 c_2}{\partial u^2}\right)^2 + \frac{178}{15}\left(\frac{\partial c_2}{\partial u}\right)^2\frac{\partial^3 c_2}{\partial u^3} + \frac{21}{20}c_2^2\frac{\partial^5 c_2}{\partial u^5}\right. \\
&\left. + \frac{259}{30}c_2\left(\frac{\partial^4 c_2}{\partial u^4}\right)\frac{\partial c_2}{\partial u} + 13\,c_2\left(\frac{\partial^2 c_2}{\partial u^2}\right)\frac{\partial^3 c_2}{\partial u^3}\right]u_{xx}u_x^2.
\end{aligned}
$$

Remarks

1. In the case c_2 =constant (KdV) the full pencil $\Pi_\lambda = P_2 - \lambda P_1$ is exact:

$$\mathrm{Lie}_e P_2 = P_1, \qquad \mathrm{Lie}_e P_1 = 0$$

where $e = \frac{\partial}{\partial u}$.

2. In the case $c_2 = u^D$:

$$\mathrm{Lie}_E P_2^{(2k)} = (kD - k - 1) P_2^{(2k)},$$

where $E = u\frac{\partial}{\partial u}$.

Local multivectors and Schouten bracket

Λ^j_{loc}= space of local multivectors

$$\alpha^{i_1,\dots,i_j} = \sum_{l_2,\dots,l_j} A^{i_1,\dots,i_j}_{l_2,\dots,l_j}(u(x_1), u_{x_1}, \dots)\delta^{(l_2)}(x_1 - x_2)\dots\delta^{(l_j)}(x_1 - x_j)$$

Associated superfunctional

$$\hat{\alpha} = \int \sum_{l_2,\dots,l_j} A^{i_1,\dots,i_j}_{l_2,\dots,l_j}(u(x_1), u_{x_1}, \dots)\theta_{i_1}\theta^{(l_2)}_{i_2}\dots\theta^{(l_j)}_{i_k}\, dx$$

Schouten bracket

$$[\alpha, \beta] := \{\hat{\alpha}, \hat{\beta}\} = \int \left(\frac{\delta\hat{\alpha}}{\delta\theta^i}\frac{\delta\hat{\beta}}{\delta u^i} + (-1)^{|\alpha|}\frac{\delta\hat{\alpha}}{\delta u^i}\frac{\delta\hat{\beta}}{\delta\theta^i}\right) dx$$

where $|\alpha|$ is the parity of α.
A local bivector Π is called *Poisson bivector* iff $[\Pi, \Pi] = 0$.

Poisson bivectors of hydrodynamic type

PBHT (Dubrovin-Novikov):

$$\omega^{ij} = g^{ij}\delta'(x-y) - g^{il}\Gamma^j_{lk}u^k_x\delta(x-y)$$

- g is a flat metric.
- Γ^j_{lk} are the Christoffel symbols of the associated Levi-Civita connection.

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Deformations of PBHT

A local bivector

$$P = \omega + \epsilon P^{(1)} + \epsilon^2 P^{(2)} + \dots + \epsilon^l P^{(l)} + \dots,$$

is called a deformation of ω if

$$[P, P] = 0, \qquad \deg P^{(l)} = l + 2$$

Degree:

- $\deg f(u) = 0$
- ∂_x increases the degrees by one
- $\deg \delta(x - y) = 1$

This implies:
$(P^{(l)})^{ij} = \sum_{s=0}^{l+1} A_k^{ij}(u, u_x, \dots)\delta^{(l+1-k)}(x - y), \qquad \deg A_k^{ij} = k$

Triviality of deformations and Poisson-Lichnerowicz cohomology

Theorem

All deformations are trivial: there exists a Miura transformation

$$u^i \to \tilde{u}^i = F_0^i(u) + \sum_k \epsilon^k F_k^i(u, u_x, \dots), \qquad \deg F_k^i = k$$

reducing $P = \omega + \epsilon P^{(1)} + \dots$ *to its dispersionless limit* ω.

The proof is by induction using

Theorem

$H^j(\omega) = 0$ *for all positive integers (Getzler in the general case, Degiovanni-Magri-Sciacca, Dubrovin-Zhang in the case* $j = 1, 2$*).*

where

$$H^j(\omega) := \frac{\ker\{d_\omega : \Lambda^j_{\text{loc}} \to \Lambda^{j+1}_{\text{loc}}\}}{\text{im}\{d_\omega : \Lambda^{j-1}_{\text{loc}} \to \Lambda^j_{\text{loc}}\}}, \qquad d_\omega := [\omega, \cdot]$$

Poisson pencils of hydrodynamic type

$$\omega_\lambda^{ij} = \omega_2^{ij} - \lambda\omega_1^{ij} = g_2^{ij}(u)\delta'(x-y) + \Gamma^{ij}_{(2)k}u_x^k\delta(x-y) + \\ -\lambda\left(g_1^{ij}(u)\delta'(x-y) + \Gamma^{ij}_{(1)k}u_x^k\delta(x-y)\right)$$

- $g_\lambda^{ij} = g_2^{ij} - \lambda g_1^{ij}$ is flat $\forall\lambda$.
- $\Gamma^{ij}_{(\lambda)k} = \Gamma^{ij}_{(2)k} - \lambda\Gamma^{ij}_{(1)k}$
- $d_{\omega_1}d_{\omega_2} + d_{\omega_2}d_{\omega_1} = 0, \qquad (d_{\omega_1} = [\omega_1, \cdot],\ d_{\omega_2} = [\omega_2, \cdot])$

Semisimplicity assumption: the roots $(r^1, \dots, r^n)$ of the equation $\det g_\lambda = 0$ are functional independent. They are called *canonical coordinates*. In canonical coordinates

$$g_1^{ij} = f^i(r^1, \dots, r^n)\delta^i_j, \qquad g_2^{ij} = r^i f^i(r^1, \dots, r^n)\delta^i_j.$$

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Central invariants

Deformations of semisimple Poisson pencils of hydrodynamic type

$$\Pi^{ij}_\lambda = \omega_\lambda + \epsilon\left[P^{ij}_\lambda \delta''(x-y) + \cdots\right] + \epsilon^2\left[Q^{ij}_\lambda \delta'''(x-y) + \cdots\right] + \mathcal{O}(\epsilon^3)$$

where $P^{ij}_\lambda = P^{ij}_2 - \lambda P^{ij}_1$ and $Q^{ij}_\lambda = Q^{ij}_2 - \lambda Q^{ij}_1$.

Central invariants:

$$c_i(r^i) = -\frac{1}{3\, f^i}\,\mathrm{Res}_{\lambda=r^i}\mathrm{Tr}\, g_\lambda^{-1} A_\lambda$$

where the tensor A^{ij}_λ is defined by

$$A^{ij}_\lambda = Q^{ij}_\lambda + (g_\lambda^{-1})_{lk} P^{li}_\lambda P^{kj}_\lambda .$$

Theorem

Two deformations of the same Poisson pencil of hydrodynamic type are Miura equivalent iff they have the same central invariants (Dubrovin-Liu-Zhang).

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Example 1: AKNS

$$\omega_2 + \epsilon P_2^{(1)} - \lambda\omega_1 = \begin{pmatrix} (2u\partial_x + u_x)\delta & v\delta' \\ \partial_x(v\delta) & -2\delta' \end{pmatrix} + \epsilon \begin{pmatrix} 0 & -\delta'' \\ \delta'' & 0 \end{pmatrix} - \lambda \begin{pmatrix} 0 & \delta' \\ \delta' & 0 \end{pmatrix}$$

In this case we have

$$g_\lambda = \begin{pmatrix} 2u & v-\lambda \\ v-\lambda & -2 \end{pmatrix}, \qquad A_\lambda = \frac{g_\lambda}{\det g_\lambda}$$

Canonical coordinates: $r^1 = v + \sqrt{-4u}$, $r^2 = v - \sqrt{-4u}$
Diagonal components of g_1: $f^1 = \frac{8}{r_2 - r_1}$, $f^2 = \frac{8}{r_1 - r_2}$

$$\begin{aligned} c_1 &= -\frac{1}{3f^1}\mathrm{Res}_{\lambda=r^1}\mathrm{Tr}\, g_\lambda^{-1}A_\lambda = -\frac{1}{3f^1}\mathrm{Res}_{\lambda=r^1}\frac{2}{\det g_\lambda} = -\frac{1}{12} \\ c_2 &= -\frac{1}{3f^2}\mathrm{Res}_{\lambda=r^2}\mathrm{Tr}\, g_\lambda^{-1}A_\lambda = -\frac{1}{3f^2}\mathrm{Res}_{\lambda=r^2}\frac{2}{\det g_\lambda} = -\frac{1}{12} \end{aligned}$$

Two component CH (Chen-Liu-Zhang, Falqui)

.

$$P_\lambda = \begin{pmatrix} (2u\partial_x + u_x)\delta & v\delta' \\ \partial_x(v\delta) & -2\delta' \end{pmatrix} - \lambda \begin{pmatrix} 0 & \delta' - \epsilon\delta'' \\ \delta' + \epsilon\delta'' & 0 \end{pmatrix}$$

$$g_\lambda = \begin{pmatrix} 2u & v-\lambda \\ v-\lambda & -2 \end{pmatrix}, \qquad A_\lambda = \frac{\lambda^2 g_\lambda}{\det g_\lambda}$$

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Exact Poisson pencil

The Poisson pencil

$$\Pi_\lambda = P_2 - \lambda P_1$$

is exact iff there exists a vector field Z, called *Liouville* vector field, such that

- $\mathrm{Lie}_Z P_2 = P_1$
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Exact Poisson pencil of hydrodynamic type

Theorem

A semisimple Poisson pencil of hydrodynamic type is exact if and only if the condition

$$\sum_{k=1}^{n} \frac{\partial f^i}{\partial r^k} = 0.$$

is satisfied.
Moreover, in canonical coordinates all the components of the vector field Z are equal to 1 (i.e. $Z = e$).

Consequences of exactness:

- $\mathrm{Lie}_e\, d_{\omega_1} - d_{\omega_1} \mathrm{Lie}_e = 0$
- $\mathrm{Lie}_e\, d_{\omega_2} - d_{\omega_2} \mathrm{Lie}_e = d_{\omega_1}$
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- $\mathrm{Lie}_e\, d_{\omega_1} d_{\omega_2} - d_{\omega_1} d_{\omega_2}\, \mathrm{Lie}_e = 0$

Costancy of the central invariants = exactness

Theorem

Let

$$\Pi_\lambda = P_2 - \lambda P_1 = \omega_2 + \sum_{k=1}^{\infty} \epsilon^{2k} P_2^{(2k)} - \lambda \left(\omega_1 + \sum_{k=1}^{\infty} \epsilon^{2k} P_1^{(2k)} \right).$$

be a Poisson pencil whose dispersionless limit $\omega_2 - \lambda\omega_1$ is semisimple and exact. Then its central invariants are constant if and only if it is exact.

Costancy of the central invariants implies exactness

Step 1: reduce Π_λ to the "standard form"

$$\Pi_\lambda = \omega_\lambda + \epsilon^2 \mathrm{Lie}_{X_{(c_1,\dots,c_n)}}\omega_1 + \sum_{k=2}^{\infty}\epsilon^{2k}P_2^{(2k)}$$

where $X_{(c_1,\dots,c_n)} = d_{\omega_2}K + d_{\omega_1}H$ with

$$K = \sum_{i=1}^{n}\int c^i(r^i)r^i_x \log r^i_x \, dx, \qquad H = -\sum_{i=1}^{n}\int r^i c^i(r^i) r^i_x \log r^i_x \, dx$$

Step 2: show that $\mathrm{Lie}_e P_2^{(2)} = 0$. Indeed:

$$\mathrm{Lie}_e \mathrm{Lie}_{X_{(c_1,\dots,c_n)}}\omega_1 = \mathrm{Lie}_{[e, X_{(c_1,\dots,c_n)}]}\omega_1 = 0$$

since $[e, X_{(c_1,\dots,c_n)}] = X_{(\frac{\partial c_1}{\partial r^1},\dots,\frac{\partial c_n}{\partial r^n})}$.

Step 3: to construct a Miura transformation

$$\Pi_\lambda \to \tilde{\Pi}_\lambda = \omega_\lambda + \sum_{k=1}^{\infty}\epsilon^{2k}\tilde{P}_2^{(2k)}, \qquad \mathrm{Lie}_e \tilde{P}_2^{(2k)} = 0, \qquad k = 1, 2, \dots$$

Costancy of the central invariants implies exactness

Step 1: reduce Π_λ to the "standard form"

$$\Pi_\lambda = \omega_\lambda + \epsilon^2 \mathrm{Lie}_{X_{(c_1,\dots,c_n)}}\omega_1 + \sum_{k=2}^{\infty} \epsilon^{2k} P_2^{(2k)}$$

where $X_{(c_1,\dots,c_n)} = d_{\omega_2}K + d_{\omega_1}H$ with

$$K = \sum_{i=1}^{n} \int c^i(r^i) r^i_x \log r^i_x \, dx, \qquad H = -\sum_{i=1}^{n} \int r^i c^i(r^i) r^i_x \log r^i_x \, dx$$

Step 2: show that $\mathrm{Lie}_e P_2^{(2)} = 0$. Indeed:

$$\mathrm{Lie}_e \mathrm{Lie}_{X_{(c_1,\dots,c_n)}}\omega_1 = \mathrm{Lie}_{[e, X_{(c_1,\dots,c_n)}]}\omega_1 = 0$$

since $[e, X_{(c_1,\dots,c_n)}] = X_{(\frac{\partial c_1}{\partial r^1},\dots,\frac{\partial c_n}{\partial r^n})}$.

Step 3: to construct a Miura transformation

$$\Pi_\lambda \to \tilde{\Pi}_\lambda = \omega_\lambda + \sum_{k=1}^{\infty} \epsilon^{2k} \tilde{P}_2^{(2k)}, \qquad \mathrm{Lie}_e \tilde{P}_2^{(2k)} = 0, \qquad k = 1, 2, \dots$$

Exactness implies the costancy of the central invariants

Step 1: reduce Π_λ to the form

$$\Pi_\lambda = \omega_\lambda + \epsilon^2 \left(\mathrm{Lie}_{X_{(c_1,\dots,c_n)}}\omega_1 + d_{\omega_2} d_{\omega_1}\tilde{H}\right) + \sum_{k=2}^{\infty} \epsilon^{2k} P_2^{(2k)}$$

with $\mathrm{Lie}_e \Pi_\lambda = \omega_1$

Step 2: the last condition implies

$$\mathrm{Lie}_e \left(\mathrm{Lie}_{X_{(c_1,\dots,c_n)}}\omega_1 + d_{\omega_2} d_{\omega_1}\tilde{H}\right) = d_{\omega_1}\left(X_{\left(\frac{\partial c_1}{\partial u^1},\dots,\frac{\partial c_n}{\partial u^n}\right)} - d_{\omega_2}(\mathrm{Lie}_e \tilde{H})\right) = 0.$$

Then there exists $\tilde{K}$ s.t.

$$X_{\left(\frac{\partial c_1}{\partial u^1},\dots,\frac{\partial c_n}{\partial u^n}\right)} = d_{\omega_2}(\mathrm{Lie}_e \tilde{H}) + d_{\omega_1}\tilde{K}.$$

The above identity makes sense only if c^i=constant.

Non constant central invariants

Theorem

Let Π_λ a Poisson pencil with polynomial central invariants of maximal degree $n-1$ and suppose that its dispersionless limit ω_λ is exact. Then there exists a Miura transformation reducing the pencil to the form

$$\tilde{\Pi}_\lambda = \omega_2 + \sum_{k=1}^{\infty} \epsilon^{2k} \tilde{P}_2^{(2k)} - \lambda\omega_1$$

with

$$\mathrm{Lie}_e^{nk-k+1} \tilde{P}_2^{(2k)} = 0, \quad k = 1, 2, \ldots .$$

Homogeneous Poisson pencils I

Euler vector field $E=\sum_{i=1}^n r^i\frac{\partial}{\partial r^i}$

$$\begin{aligned}\mathrm{Lie}_E\omega_1 &= (d-2)\omega_1\\ \mathrm{Lie}_E\omega_2 &= (d-1)\omega_2.\end{aligned}$$

Consequences

- $\mathrm{Lie}_E d_{\omega_1} - d_{\omega_1}\mathrm{Lie}_E = (d-2)d_{\omega_1}$
- $\mathrm{Lie}_E d_{\omega_2} - d_{\omega_2}\mathrm{Lie}_E = (d-1)d_{\omega_2}$
- $\mathrm{Lie}_E d_{\omega_1}\, d_{\omega_2} - d_{\omega_1}\, d_{\omega_2}\mathrm{Lie}_E = (2d-3)d_{\omega_1}\, d_{\omega_2}$

Homogeneous Poisson pencils II

Theorem

Let Π_λ be a homogeneous Poisson pencil. Suppose that the central invariants are homogeneous functions of degree D. Then there exists a Miura transformation reducing Π_λ to the form

$$\tilde{\Pi}_\lambda = \omega_2 + \sum_{k=1}^{\infty} \epsilon^{2k} \tilde{P}_2^{(2k)} - \lambda \omega_1$$

with

$$\mathrm{Lie}_E \tilde{P}_2^{(2k)} = [(k+1)(d-1) + kD] \tilde{P}_2^{(2k)}, \qquad k = 1, 2, \ldots$$