Singular polynomials from Frobenius manifolds

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Frobenius manifolds

Origins:

- (dispersionless) integrable systems
- topological field theories
- enumerative geometry
- singularity theory

Frobenius manifold [Dubrovin] is a manifold M with the following structures:

- M has a flat metric η
- $\forall x \in M$ multiplication $m : T_xM \times T_xM \to T_xM$ is defined, m is associative and commutative with covariantly constant identity e
- T_xM is a Frobenius algebra: $\forall u, v, w \in T_xM$ one has $\eta(u \cdot v, w) = \eta(u, v \cdot w)$
- Existence of prepotential
  Let $t^1, \ldots, t^n$ be flat coordinates (that is $\eta = const$, $n = dimM$).
  Let $c_{\alpha\beta}^{\gamma}$ be structure constants for m: $\partial_{t^\alpha} \cdot \partial_{t^\beta} = c_{\alpha\beta}^{\gamma}\partial_{t^\gamma}$.
  Let $c_{\alpha\beta\gamma} = c_{\alpha\beta}^{\varepsilon}\eta_{\varepsilon\gamma}$. Then $\exists F(t^1, \ldots, t^n)$ s.t. $c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$.
- Quasihomogeneity
  There exists an Euler vector field E s.t. $\mathcal{L}_E F = dF +$ quadratic terms for some $d \in \mathbb{R}$

Associativity equations

Note that associativity is a system of PDEs on F:

$$c_{\alpha\beta}^{\gamma} c_{\gamma\varepsilon}^{\delta} = c_{\beta\varepsilon}^{\gamma} c_{\gamma\alpha}^{\delta}, \quad \forall \alpha, \beta, \gamma, \varepsilon, \delta = 1, \ldots, n,$$

or, equivalently, $c_{\alpha\beta\tilde{\gamma}} \eta^{\tilde{\gamma}\gamma} c_{\gamma\varepsilon\delta} = c_{\beta\varepsilon\tilde{\gamma}} \eta^{\tilde{\gamma}\gamma} c_{\gamma\alpha\delta}$. These are the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations.

Dubrovin's almost duality

Let M be a Frobenius manifold, $\cdot$ - multiplication, η - flat metric, E-Euler field.
Define M^* - almost Frobenius manifold, with $\star$ - multiplication, g - flat metric:

- $M^* = M \setminus \Sigma$, Σ - some discriminant subset
- $u \star v = E^{-1} \cdot u \cdot v, \quad \forall u, v \in T_x M$
- $g(\omega_1, \omega_2) = i_E(\omega_1 \cdot \omega_2) \quad \forall \omega_1, \omega_2 \in T_x^* M$,
  where multiplication of 1-forms is defined by $dt^\alpha \cdot dt^\beta = \eta^{\delta\beta} c^\alpha_{\gamma\delta} dt^\gamma$

Note that E is the identity field for M^*, it is not covariantly constant for the Levi-Cevita connection ∇^g of the metric g, so M^* is not a Frobenius manifold.

Root system and Coxeter group

Let $V = \mathbb{C}^n$, let $g(u,v) = (u,v)$ be the standard bilinear form in V. Let $\mathcal{R} \subset V$ be a Coxeter root system. That is

- $\forall \alpha \in \mathcal{R} \quad s_\alpha \mathcal{R} = \mathcal{R}$,
  where s_α is orthogonal reflection with respect to the hyperplane $(\alpha, x) = 0$.
- If $\alpha, \beta \in \mathcal{R}$ and $\alpha \sim \beta$ then $\alpha = \pm\beta$.

We suppose also that $\mathcal{R}$ is irreducible, $(\alpha,\alpha) = 2 \ \forall \alpha \in \mathcal{R}$ and that $rank(\mathcal{R}) = n$.
Let $W = \langle s_\alpha | \alpha \in \mathcal{R} \rangle$ be the corresponding Coxeter group.

Space of orbits

By Chevalley's theorem $\mathbb{C}[x_1,\ldots,x_n]^W \cong \mathbb{C}[y_1,\ldots,y_n]$, where y_i are homogeneous polynomials, $\deg y_i(x) = d_i$ - degrees of the Coxeter group.
We order $d_1 > d_2 \geq \ldots > d_n = 2$, $d_1 = h$ - Coxeter number of W.
$y_1,\ldots,y_n$ are coordinates on the orbit space $M = V/W \approx \mathbb{C}^n$
Define Saito metric $\eta^{\cdot\cdot}(y) = \partial_{y_1} g^{\cdot\cdot}(y)$, it is flat [Saito, Yano, Sekiguchi'80].
Fix Saito flat coordinates $t^1,\ldots,t^n \in \mathbb{C}[x_1,\ldots,x_n]^W$, $\deg t^\alpha = d_\alpha$, by normalization $\eta^{\alpha\beta} = \delta^{\alpha+\beta,n+1}$ $(1 \leq \alpha,\beta \leq n)$.
Let $E = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} = \sum_{i=1}^n d_i y_i \frac{\partial}{\partial y_i}$ – Euler vector filed. Identity field $e = \frac{\partial}{\partial y_1}$.

Theorem (Dubrovin'94, 04)

M is a Frobenius manifold; prepotential F is polynomial in the Saito coordinates.
The almost dual prepotential $F^* = \frac{1}{2}\sum_{\alpha\in\mathcal{R}_+}(\alpha, x)^2\log(\alpha, x)$.

Remark

Saito flat coordinates are known explicitly for any Coxeter group W:

- *[Saito, Yano, Sekiguchi'80], $W \neq E_7, E_8$*
- *[Abriani/Talamini'09/10], $W = E_7, E_8$*

Solutions to WDVV equations of type $F^* = \sum_{\alpha \in \mathcal{A}} (\alpha, x)^2 \log(\alpha, x)$ have been studied:

- Seiberg-Witten theory, [Marshakov, Mironov, Morozov] 1997
- $\mathcal{A}$ is a root system [Martini-Gragert'99]
- Geometrical reformulation of WDVV as $\vee$-systems [Veselov]; n-parameter deformations of classical Coxeter rank n root systems [Chalykh, V] 1999-2001
- Closure under restriction and taking subsystems, almost duality on the discriminant strata, solutions from superalgebras Lie, rank 3 study [F, Veselov] 2006-07
- Relation of WDVV and superconformal mechanics [Wyllard'00; Bellucci, Galajinsky, Latini'05]
- Superconformal extensions for particular solutions of WDVV of this type [Galajinsky,Krivonos, Lechtenfeld, Polovnikov] 2008-2011
- rank 3 solutions with small number of vectors [Lectenfeld, Schwerdtfeger, Thrigen] 2011

Saito coordinates in type A

Example

Let $\mathcal{R} = \mathcal{A}_n \subset \mathbb{C}^{n+1}$ be given in its standard embedding:

- $V \subset \mathbb{C}^{n+1}$ is defined by $\sum_{i=1}^{n+1} z_i = 0$ where z_i are standard coordinates in $\mathbb{C}^{n+1}$
- $\mathcal{R} = \{\pm(e_i - e_j) | 1 \leq i < j \leq n+1\} \subset V$

Define

$$s^\alpha = Res_{z=\infty} \prod_{i=1}^{n+1} (z - z_i)^{\frac{\alpha}{n+1}} dz = Res_{u=0} \frac{\prod_{i=1}^{n+1} (1 - z_i u)^{\frac{\alpha}{n+1}}}{u^{\alpha+2}}$$

for $\alpha = 1, \ldots, n$.

Then $t^\alpha = \kappa_\alpha s^\alpha|_V$ for some $\kappa_\alpha \in \mathbb{R}$.

For $n = 3$, $F = \frac{(t^1)^2 t^3 + t^1 (t^2)^2}{2} + \frac{(t^2)^2 (t^3)^2}{4} + \frac{(t^3)^5}{60}$.

Twisted periods

Definition (Dubrovin'04)

Fix $c \in \mathbb{C}$. A locally defined function $Q = Q(x)$ is called twisted period if the following system of Gauss-Manin equations holds:

$$\frac{\partial^2 Q}{\partial x_i \partial x_j} = c \sum_{k=1}^{n} \frac{\partial^3 F^*}{\partial x_i \partial x_j \partial x_k} \frac{\partial Q}{\partial x_k}, \quad 1 \leq i, j \leq n.$$

Equivalently, twisted periods Q satisfy $\nabla^{g,c}(dQ) = 0$ where $\nabla^{g,c}$ is the flat symmetric Dubrovin connection on M^*, namely, for any $u, v \in TM^*$ $\nabla_u^{g,c} v = \nabla_u^g v + cu * v$.

Locally there exist n independent twisted periods. We are interested in twisted periods Q which are W-invariant polynomials. They exist for special values of the parameter c only.

Observation: The Saito polynomial $Q = t^\beta$, $1 \le \beta \le n$ is a twisted period with $c = \frac{d_\beta - 1}{h}$.
Generalisation: Let $m \in \mathbb{Z}_+$, choose β, $1 \le \beta \le n$. Define

$$Q = \sum_{\alpha=1}^{n} d_\alpha \zeta_\alpha^{(m)} t^\alpha, \qquad (1)$$

where $\zeta_\alpha^{(m)}$ is the α-th component of the covector $\zeta^{(m)} = \zeta^{(0)} \prod_{j=1}^{m} U(\frac{d_\beta - 1}{h} + j + \Lambda)^{-1}$, where $\zeta^{(0)} = (0, \dots, 0, 1, 0, \dots, 0)$ (1 on the β-th place), $\Lambda = Diag(\frac{1-d_1}{h}, \dots, \frac{1-d_n}{h})$, and U is the matrix $U_\alpha^\beta = g^{\beta\gamma}\eta_{\gamma\alpha} = g^{\beta, n+1-\alpha} = \sum_{a=1}^{n} \frac{\partial t^\beta}{\partial x^a} \frac{\partial t^{n+1-\alpha}}{\partial x^a}$.

Theorem (F., Silantyev'11)

Let $Q \in \mathbb{C}[x_1, \dots, x_n]^W$ be a twisted period. Then $c = \frac{d_\beta - 1}{h} + m$ for some $m \in \mathbb{Z}_+$ and some degree d_β of W. Further, Q is given by a multiple of (1) (or by a linear combination of polynomials (1) corresponding to the same c if W has multiple degrees).

Dunkl operators

Let $c \in \mathbb{C}$, $\xi \in V$. Define $\nabla_\xi = \partial_\xi - c\sum_{\alpha\in\mathcal{R}_+} \frac{(\alpha,\xi)}{(\alpha,x)}(1-s_\alpha)$.
Then $[\nabla_\xi, \nabla_\eta] = 0$ for any $\xi, \eta \in V$ [Dunkl'89]
Note that $\nabla_\xi : \mathbb{C}[x] \to \mathbb{C}[x]$, where $\mathbb{C}[x] \equiv \mathbb{C}[x_1, \dots, x_n]$

Definition (Dunkl, de Jeu, Opdam'94)

A polynomial $p(x)$ is called singular if $\nabla_\xi p(x) = 0$ for any $\xi \in V$.

Define the rational Cherednik algebra [Etingof,Ginzburg'02]
$H_c(W) = \langle \mathbb{C}[x], \mathbb{C}[\nabla], \mathbb{C}W \rangle$
Then $\mathbb{C}[x]$ is a module for $H_c(W)$; it is irreducible for generic c.
$\mathbb{C}[x]$ is reducible if and only if $c - \frac{j}{d_i} \in \mathbb{N}$ for some degree d_i and for some $j, 1 \le j \le d_i - 1$ [DJO].

Singular polynomials and $H_c(W)$-modules

Singular polynomials correspond to submodules of $\mathbb{C}[x]$:

- Let $M \subset \mathbb{C}[x]$ be $H_c(W)$-invariant. Then $M = \oplus_{i \geq 0} M_i$, where M_i consists of homogeneous polynomials of degree i. Let k be such that $M_i = 0$ for $i < k$, and $M_k \neq 0$. Then M_k consists of singular polynomials.
- On the other hand, if $p(x)$ is singular then $N = \langle wq(x)p(x) | w \in W, q(x) \in \mathbb{C}[x] \rangle$ is a non-trivial submodule of $\mathbb{C}[x]$.

In type A all singular polynomials are known [Dunkl'05]
In type A the spaces N provide all submodules of $\mathbb{C}[x]$ [Etingof, Stoica'09]

Saito polynomials give singular polynomials

Proposition (F., Silantyev'11)

$q(x) = \partial_\xi t^\beta$ is singular with $c = \frac{d_\beta - 1}{h}$ $\quad \forall \xi \in V, \beta = 1, \ldots, n$.

Proposition (F., Silantyev'11)

For any $\xi \in V$, $\beta = 1, \ldots, n$ the polynomial

$$q(x) = \sum_{a,\alpha=1}^{n} \frac{1}{d_\beta - d_\alpha + h} \frac{\partial t^\beta}{\partial x^a} \frac{\partial t^{n+1-\alpha}}{\partial x^a} \partial_\xi t^\alpha$$

is singular with the parameter $c = \frac{d_\beta - 1}{h} + 1$.

Further shifting

Let $m \in \mathbb{Z}_+$, choose β, $1 \leq \beta \leq n$, and $\xi \in V$.
Define $q(x) = \sum_{\alpha=1}^{n} \zeta_\alpha^{(m)} \partial_\xi t^\alpha$, where $\zeta_\alpha^{(m)}$ is the α-th component of the covector
$\zeta^{(m)} = \zeta^{(0)} \prod_{j=1}^{m} U(\frac{d_\beta - 1}{h} + j + \Lambda)^{-1}$,
where $\zeta^{(0)} = (0, \dots, 0, 1, 0, \dots, 0) \quad$ (1 on the β-th place),
$\Lambda = Diag(\frac{1-d_1}{h}, \dots, \frac{1-d_n}{h})$,
and U is the matrix $U_\alpha^\beta = g^{\beta\gamma}\eta_{\gamma\alpha} = g^{\beta, n+1-\alpha} = \sum_{a=1}^{n} \frac{\partial t^\beta}{\partial x^a} \frac{\partial t^{n+1-\alpha}}{\partial x^a}$.

Theorem (F., Silantyev'11)

$q(x)$ is singular for $c = \frac{d_\beta - 1}{h} + m$.

Theorem (F., Silantyev'11)

This construction exhausts all the singular polynomials in the isotypic component of the reflection representation. Equivalently, any such singular polynomial $q(x)$ has the form $q = \partial_\xi Q$ where $Q \in \mathbb{C}[x_1, \ldots, x_n]^W$ is a twisted period, for some $\xi \in V$.

Connection to Calogero-Moser systems

Take $L = \sum_{i=1}^{n} \nabla_{e_i}^2$.
Then $L|_{C[x]^W} = \Delta - \sum_{\alpha \in \mathcal{R}_+} \frac{2c}{(\alpha,x)} \partial_\alpha$ is the Calogero-Moser operator [Heckman'91]
Further $\forall \sigma \in \mathbb{C}[x]^W$

$$L_\sigma = \sum_{i=1}^{n} \sigma(\nabla_{e_i}) : \mathbb{C}[x]^W \to \mathbb{C}[x]^W$$

is the quantum integral for L: $[L, L_\sigma] = 0$.
More generally, $[L_\sigma, L_\tau] = 0 \ \forall \sigma, \tau \in \mathbb{C}[x]^W$, and $L = L_{\sum x_i^2}$.

Proposition (F., Silantyev'11)

The Saito polynomials t^β satisfy $L_\sigma t^\beta = 0 \qquad \forall \sigma \in \mathbb{C}[x]^W$ if $c = (d_\beta - 1)/h$. More generally, $L_\sigma Q = 0$ if $c = (d_\beta - 1)/h + m$.

Bosonic potential in supersymmetric mechanics

We have $q(x) = \partial_\xi Q(x)$ where $Q = \sum_{\alpha=1}^n d_\alpha \zeta_\alpha^{(m)} t^\alpha$.
For example, $Q = t^\beta$ when $m = 0$ (i.e. $c = \frac{d_\beta - 1}{h}$, $\beta = 1, \dots, n$).
This gives $D(2,1;\alpha)$ superconformal mechanical system with bosonic potential V_B proportional to

$$\frac{1}{Q^2}\sum_{i=1}^n \left(\frac{\partial Q}{\partial x^i}\right)^2$$

at $\alpha = -\frac{1}{2}(d_\beta + hm)$.
Conformal mechanics: $H = \frac{1}{2}\sum p_i^2 + V_B(x)$ is a part of the $so(1,2)$ conformal algebra, $[D,H] = -iH, [H,K] = 2iD, [D,K] = iK$, where $D = -\frac{1}{4}(x^i p_i + p_i x^i)$, $K = \frac{1}{2}x^i x^i$ (here $[x^i, p_j] = i\delta^i_j$).

There is a $\mathcal{N}=4$ supersymmetric extension of the symmetry algebra $so(1,2)$ to the superalgebra $D(2,1;\alpha)$ where H is replaced with

$$\widetilde{H}=H+V_{\psi}.$$

The additional part of the potential V_{ψ} depends on $4n$ fermionic variables. It is defined in terms of the solution F^* of WDVV equations and in terms of the twisted period Q (c.f. [Krivonos, Lectenfeld, 2010])

Sketch of proofs

1. If $q(x)$ is singular and $q(x)$ generates a reflection representation of the Coxeter group W then $q = \partial_\xi Q$ for some $Q \in \mathbb{C}[x]^W$, and for some vector $\xi \in V$.
2. Let $Q \in \mathbb{C}[x]^W$. Then $\partial_\xi Q$ is singular if and only if Q is a twisted period:

$$\frac{\partial^2 Q}{\partial x_i \partial x_j} = c \sum_{k=1}^{n} \frac{\partial^3 F^*}{\partial x_i \partial x_j \partial x_k} \frac{\partial Q}{\partial x_k} = c \sum_{\substack{k=1 \\ \gamma \in \mathcal{R}_+}}^{n} \frac{\gamma_i \gamma_j \gamma_k}{(\gamma, x)} \frac{\partial Q}{\partial x_k}, \quad 1 \le i, j \le n$$

3.[Dubrovin'04] Q is a twisted period if and only if the following equations hold

$$\partial_{t^\alpha} \zeta(t)\, U = \zeta(t)\big(c + \Lambda\big) c_\alpha, \quad 1 \le \alpha \le n, \tag{2}$$

where $\zeta(t) = (\partial_{t^1} Q(t), \ldots, \partial_{t^n} Q(t))$, and c_α is the matrix $(c_\alpha)^\gamma_\beta = c^\gamma_{\alpha\beta}$ - the tensor of structure constants in the flat coordinates.

4. [Dubrovin'04] If $\zeta(t)$ is a solution of (2) then $\partial_{t^1}\zeta(t)$ is also a solution of (2) with c replaced by $c-1$. Thus if Q is a twisted period with parameter c then $\partial_{t^1}Q$ is a twisted period with parameter $c-1$.
5. $Q = t^\beta$ satisfies (2) if $c = \frac{d_\beta - 1}{h}$.
6. Let $\zeta(t)$ be a solution of the system (2). Assume that $c \neq \dfrac{d_\alpha - 1}{h} - 1$ for all $\alpha = 1, \ldots, n$. Then

$$\widehat{\zeta}(t) = \zeta(t) U \big(c + 1 + \Lambda\big)^{-1}$$

is a solution of the system (2) with c replaced by $c+1$.

Thank you for your attention !