towards Lax formulation of integrable hierarchies of topological type

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With:
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- Dubrovin-Zhang on Principal Hierarchies

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Using:

- Hirota biliner/quadratic equation
- Wave function and Sato-Wilson equation

KdV hierarchy, Lax equation:

$$\frac{\partial L}{\partial q_n} = \frac{a_n}{\sqrt{\hbar}} \left[\left(L^{n+\frac{1}{2}} \right)_+ , L \right] , \qquad n = 0, 1, 2, \dots ,$$

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$$\begin{aligned} L^{\frac{1}{2}} = & \sqrt{\hbar}\partial + \frac{u}{\sqrt{\hbar}}\partial^{-1} - \frac{u'}{2\sqrt{\hbar}}\partial^{-2} \\ & + \frac{\hbar u'' - 2u^2}{4\hbar^{3/2}}\partial^{-3} - \frac{\hbar u^{(3)} - 12uu'}{8\hbar^{3/2}}\partial^{-4} + \cdots . \end{aligned}$$

Lax equations are compatibility equations of the wave function ψ

$$L\psi(x, q_i, \lambda) = \lambda^2 \psi(x, q_i, \lambda)\,,$$
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Hirota bilinear/quadratic equation for tau function ($x = x' = 0$):

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$$\begin{aligned}\psi(x, q_i, \lambda) &= \frac{\Gamma(\lambda)\tau(q_i + \delta_{i0}x)}{\tau(q_i + \delta_{i0}x)} e^{\frac{x\lambda}{\sqrt{\hbar}}} \\ &= P(\lambda)\Gamma_+(\lambda)e^{\frac{x\lambda}{\sqrt{\hbar}}}\end{aligned}$$

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- Sato-Wilson equation:

$$\frac{\partial P(\sqrt{\hbar}\partial)}{\partial q_n} P(\sqrt{\hbar}\partial)^{-1} = -\frac{a_n}{\sqrt{\hbar}} \left(P(\sqrt{\hbar}\partial)(\sqrt{\hbar}\partial)^{2n+1} P(\sqrt{\hbar}\partial)^{-1}\right)_- .$$

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Note:

$$\frac{a_n}{\sqrt{\hbar}} \mathrm{Res}_\partial L^{n+\frac{1}{2}} = \frac{\partial^2 \log \tau}{\partial q_0 \partial q_n}$$

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and $\tau(q) = \prod_\gamma^N \tau_\gamma(q_{\gamma 0}, q_{\gamma 1}, \ldots)$
then for every $1 \le \alpha \le N$:

$$\mathrm{Res}_{\lambda_\alpha} \lambda_\alpha^{2p} \Gamma_\alpha(\lambda_\alpha)\tau(q) \otimes \Gamma_\alpha(-\lambda_\alpha)\tau(q') = 0$$

where

$$\Gamma_{\alpha,-}(\lambda_\alpha) = \exp\left(-\sqrt{\hbar}\sum_{n=0}^{\infty} b_n \lambda_\alpha^{-2n-1} \frac{\partial}{\partial q_{\alpha,n}}\right),$$
$$\Gamma_{\alpha,+}(\lambda_\alpha) = \exp\left(\frac{1}{\sqrt{\hbar}}\sum_{n=0}^{\infty} a_n q_{\alpha,n} \lambda_\alpha^{2n+1}\right)$$

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$$\hat{r} = \sum_{\alpha,\beta=1}^{N} (r_\ell)_{\alpha\beta} \left(\sum_{i=0}^{\infty} q_{\alpha,i} \frac{\partial}{\partial q_{\beta,\ell+i}} + \frac{\hbar}{2} \sum_{i+j=\ell-1} (-1)^{i+1} \frac{\partial^2}{\partial q_{\alpha,i} \partial q_{\beta,j}} \right),$$

$$\hat{s} = \sum_{\alpha,\beta=1}^{N} (s_\ell)_{\alpha\beta} \left(\sum_{i=0}^{\infty} q_{\alpha,i+\ell} \frac{\partial}{\partial q_{\beta,i}} - \frac{1}{2\hbar} \sum_{i+j=\ell-1} (-1)^{\ell+i} q_{\alpha,i} q_{\beta,j} \right).$$

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Let this $\hat{G}$ act on the Hirota bilinear identity:

$$\begin{aligned}\mathrm{Res}_{\lambda_\alpha} \lambda_\alpha^{2p} (\hat{G}\Gamma_{\alpha,+}(\lambda_\alpha)\hat{G}^{-1})(\hat{G}\Gamma_{\alpha,-}(\lambda_\alpha)\hat{G}^{-1})\hat{G}\tau(q) \\ (\hat{G}'\Gamma_{\alpha,+}(-\lambda_\alpha)\hat{G'}^{-1})(\hat{G}\Gamma_{\alpha,-}(-\lambda_\alpha)\hat{g'}^{-1})\hat{G}'\tau(q')\, d\lambda_\alpha = 0.\end{aligned}$$

Next rearange, then up to $O(\epsilon^2)$ one finds:

$$\mathrm{Res}_\lambda \lambda^{2p} \Big(G_{\alpha,+}(\lambda)\Gamma_{\alpha,+}(\lambda) G_{\alpha,-}(\lambda)\Gamma_{\alpha,-}(\lambda)\hat{G}\tau(q)$$

$$G'_{\alpha,+}(-\lambda)\Gamma_{\alpha,+}(-\lambda) G'_{\alpha,-}(-\lambda)\Gamma_{\alpha,-}(-\lambda)\hat{G}'\tau(q')\Big)\, d\lambda = 0\,.$$

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where for $G = R$:

$$R_{\alpha,+}(\lambda) := \exp\left(\epsilon (r_\ell)_{\alpha\alpha} d_\ell \lambda^{2\ell}\right) \\ \times \exp\left(\frac{\epsilon(-1)^{\ell-1}}{\sqrt{\hbar}} \sum_\beta (r_\ell)_{\alpha\beta} \left(\sum_{n\geq \ell} a_n \lambda^{2n+1} q_{\beta,n-\ell}\right)\right) \;;$$

$$R_{\alpha,-}(\lambda) := \exp\left(\epsilon\sqrt{\hbar}\sum_\beta (r_\ell)_{\alpha\beta} \sum_{n=0}^{\ell-1} (-1)^{n+1} a_n \lambda^{2n+1} \frac{\partial}{\partial q_{\beta,\ell-1-n}}\right) \\ \times \exp\left(\epsilon\sqrt{\hbar} \sum_{\beta,n>0} (r_\ell)_{\alpha\beta} b_n \lambda^{-2n-1} \frac{\partial}{\partial q_{\beta,n+\ell}}\right) .$$

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- and write the Hirota bilinear equation as:

$$\mathrm{Res}_\lambda \lambda^{2p} V_{\alpha,G}(x,q,\lambda) e^{\frac{x\lambda}{\sqrt{\hbar}}} V_{\alpha,G}(x',q',-\lambda) e^{\frac{-x'\lambda}{\sqrt{\hbar}}}\, d\lambda = 0$$

where $V_{\alpha,G}(x,q,\lambda) =$

$$G_{\alpha,-}(\lambda)(P_{\alpha,G}(\lambda)) \frac{G_{\alpha,-}(\lambda)(\hat{G}\tau(x,q))}{\hat{G}\tau(x,q)} G_{\alpha,+}(\lambda)\Gamma_{\alpha,+}(\lambda) e^{\epsilon g_{\alpha,G}(x,\lambda)}$$

and

$$P_{\alpha,G}(\lambda) = \frac{\Gamma_{\alpha,-}(\lambda)(\hat{G}\tau(x,q))}{\hat{G}\tau(x,q)}$$

Write

$$V_{\alpha,G}(x,q,\sqrt{\hbar}\partial) = \left(P_{\alpha,G}(\sqrt{\hbar}\partial) + \epsilon Q_{\alpha,G}(x,q,\sqrt{\hbar}\partial)\right)\Gamma_{\alpha,+}(\sqrt{\hbar}\partial)$$

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Define

$$L_{\alpha,G} := P_{\alpha,G}\hbar\partial^2 P_{\alpha,G}^{-1} = L_\alpha + (L_{\alpha,G})_- = L_\alpha + \epsilon[L_\alpha, Q_{\alpha,G}P_{\alpha,G}^{-1}]_-$$

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Deformed Sato-Wilson equation:

$$\frac{\partial P_{\alpha,G}}{\partial q_{\beta,n}} P_{\alpha,G}^{-1} + \delta_{\alpha\beta}\frac{a_n}{\sqrt{\hbar}}\left(L_{\alpha,G}^{n+\frac{1}{2}}\right)_- \\ + \epsilon\left(\frac{\partial Q_{\alpha,G}P_{\alpha,G}^{-1}}{\partial q_{\beta,n}} - \delta_{\alpha\beta}\frac{a_n}{\sqrt{\hbar}}\left[\left(L_\alpha^{n+\frac{1}{2}}\right)_+, Q_{\alpha,G}P_{\alpha,G}^{-1}\right]\right)_- = 0\,,$$

$$
\begin{aligned}
Q_{\alpha,R}P_{\alpha,R}^{-1} = {}& (r_\ell)_{\alpha\alpha} d_\ell L_{\alpha,R}^{\ell} + \frac{(-1)^{\ell-1}}{\sqrt{\hbar}} \sum_{\beta} (r_\ell)_{\alpha\beta} a_\ell x L_{\alpha,R}^{\ell+\frac{1}{2}} \\
& - (r_\ell)_{\alpha\alpha} \sum_{n=0}^{\ell-1} (-1)^{n+1} a_n a_{\ell-1-n} \left(L_{\alpha,R}^{\ell-n-\frac{1}{2}} \right)_{-} L_{\alpha,R}^{n+\frac{1}{2}} \\
& - (r_\ell)_{\alpha\alpha} \sum_{n\geq 0} b_n a_{n+\ell} \left(L_{\alpha,R}^{n+\ell+\frac{1}{2}} \right)_{-} L_{\alpha,R}^{-n-\frac{1}{2}} \\
& + \frac{(-1)^{\ell-1}}{\sqrt{\hbar}} \sum_{\beta} (r_\ell)_{\alpha\beta} \sum_{n\geq \ell} a_n q_{\beta,n-\ell} L_{\alpha,R}^{n+\frac{1}{2}} \\
& + \sqrt{\hbar} \sum_{\beta} (r_\ell)_{\alpha\beta} \sum_{n=0}^{\ell-1} (-1)^{n+1} a_n \frac{\partial \log \tau}{\partial q_{\beta,\ell-1-n}} L_{\alpha,R}^{n+\frac{1}{2}} \\
& + \sqrt{\hbar} \sum_{\beta} (r_\ell)_{\alpha\beta} \sum_{n\geq 0} b_n \frac{\partial \log \tau}{\partial q_{\beta,\ell+n}} L_{\alpha,R}^{-n-\frac{1}{2}}
\end{aligned}
$$

In the work of Buryak, Posthuma and Shadrin:

$$\hat{G}\Omega_{\alpha,p;\beta,q} := \hbar \frac{\partial^2 \log \hat{G}\tau}{\partial q_{\alpha,p} \partial q_{\beta,q}}$$

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Now

$$\hat{G}\Omega_{\alpha,0;\beta,p} = -\mathrm{Res}_\partial \frac{\partial P_{\alpha,G}}{\partial q_{\beta,q}} P_{\alpha,G}^{-1}$$

We show: The Res_∂ of the deformed Sato-Wilson equations equal the deformation formula's of Buryak, Posthuma and Shadrin.