The Gould-Hopper Polynomials in the Novikov-Veselov equation

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Geometrical Methods in Mathematical Physics, Moscow, Dec. 2011

Outline

The Novikov-Veselov [1984] equation is defined by (U and t is real) [Athorne, Dubrovsky, Matveev, Nimmo, Salle, Y.Ohta]

$$\begin{aligned} U_t &= \partial_z^3 U + \bar{\partial}_z^3 U - 3\partial_z(VU) - 3\bar{\partial}_z(\bar{V}U), \\ \bar{\partial}_z V &= \partial_z U. \end{aligned} \tag{1}$$

When $z = \bar{z} = x$, we get the KdV equation ($U = V = \bar{V}$)

$$U_t = 2U_{xxx} - 12UU_x$$

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The Novikov-Veselov equation can be represented as the form of Manakov's triad

$$H_t = [A, H] - BH,$$

where H is the two-dimension Schrodinger operator

$$H = \partial_z \bar{\partial}_z + U$$

and

$$A = \partial_z^3 - V\partial_z + \bar{\partial}_z^3 - \bar{V}\bar{\partial}_z, \quad B = V_z + \bar{V}_{\bar{z}}.$$

It is equivalent to the linear representation

$$H\psi = 0, \quad \partial_t \psi = A\psi. \tag{2}$$

Motives:

- Integrable deformation of Schrodinger Operator [Athorne, Matveev, Nimmo, Y.Ohta]
- D-bar dressing method [Boiti, Dubrovsky, Leon, Pempinelli, Tsai]
- Two dimensional generalization of KdV equation [S.P.Novikov, A.Veselov, L.V. Bogdanov, P.G. Grinevich]
- Two-component BKP equation [R. Hirota, I.Krichever, Takasaki, Si-Qi Liu, C.Z. Wu, Y.J. Zhang]
- The Tzitzeica equation [E.Ferapontov., A.E. Mironov]
- Integrable deformation of minimal Lagrangian tori in CP^2 [A.E. Mironov]
- Integrable deformation of Dirichlet-to-Neumann map (Electrical Impedance Tomography, [M.Lassas, J. Mueller, A.Stahel])

Let $H\psi = H\omega = 0$. Then via the Moutard transformation[1878]

$$\begin{aligned} U(z,\bar z) &\longrightarrow \hat U(z,\bar z) = U(z,\bar z) + 2\partial\bar\partial \ln[i\int(\psi\partial\omega \\ &- \omega\partial\psi)dz - (\psi\bar\partial\omega - \omega\bar\partial\psi)d\bar z], \\ \psi &\longrightarrow \theta = \frac{i}{\omega}\int(\psi\partial\omega - \omega\partial\psi)dz - (\psi\bar\partial\omega - \omega\bar\partial\psi)d\bar z \end{aligned}$$

one can construct a new Schrodinger operator

$$\hat H = \partial_z\bar\partial_z + \hat U$$

and $\hat\psi = \frac{1}{\theta}$ such that

$$\hat H\hat\psi = 0.$$

The extended Moutard transformation was established such that $\hat{U}(t,z,\bar{z})$ and $\hat{V}(t,z,\bar{z})$ defined by [Matveev, Salle, Athorne, Nimmo, H.C.Hu, S.Y.Lou and Q.P.Liu, 1991-2003]

$$\hat{U}(t,z,\bar{z}) = U(t,z,\bar{z}) + 2\partial\bar{\partial}\ln iW,$$

where

$$\begin{aligned} W &= \int(\psi\partial\omega-\omega\partial\psi)dz-(\psi\bar{\partial}\omega-\omega\bar{\partial}\psi)d\bar{z} && (3)\\ &+ [\psi\partial^3\omega-\omega\partial^3\psi+\omega\bar{\partial}^3-\psi\bar{\partial}^3\omega \\ &+ 2(\partial^2\psi\partial\omega-\partial\psi\partial^2\omega)-2(\bar{\partial}^2\psi\bar{\partial}\omega-\bar{\partial}\psi\bar{\partial}^2\omega) && (4)\\ &+ 3V(\psi\partial\omega-\omega\partial\psi)-3\bar{V}(\psi\bar{\partial}\omega-\omega\bar{\partial}\psi)]dt,\\ \hat{V} &= V+2\partial\partial\ln iW, && (5) \end{aligned}$$

will also satisfy the Novikov-Veselov equation.

In particular, we can use $U = V = 0$ as the seed solution. Then $H = \partial\bar{\partial}$. Let us consider the holomorphic functions $P(z, t)$:

$$\frac{\partial P}{\partial t} = \frac{\partial^3 P}{\partial z}. \tag{6}$$

Then we have the following

Theorem [Taimanov and Tsarev, 2008]
Let $\mathcal{P}_1(t,z)$ and $\mathcal{P}_2(t,z)$ be polynomial functions of z and satisfy (6). One defines $\omega_1 = \mathcal{P}_1 + \bar{\mathcal{P}}_1$ and $\omega_2 = \mathcal{P}_2 + \bar{\mathcal{P}}_2$. Then

$$U(t,z,\bar z) = 2\partial\bar\partial \ln iW,$$

where

$$\begin{aligned} W &= \mathcal{P}_1\bar{\mathcal{P}}_1 - \mathcal{P}_2\bar{\mathcal{P}}_2 + \int [(\mathcal{P}_1'\mathcal{P}_2 - \mathcal{P}_1\mathcal{P}_2')dz \\ &+ (\bar{\mathcal{P}}_1\bar{\mathcal{P}}_2{}' - \bar{\mathcal{P}}_1{}'\bar{\mathcal{P}}_2)d\bar z] \\ &+ \int [\mathcal{P}_1'''\mathcal{P}_2 - \mathcal{P}_1\mathcal{P}_2''' + 2(\mathcal{P}_1'\mathcal{P}_2'' - \mathcal{P}_1''\mathcal{P}_2') + \bar{\mathcal{P}}_1\bar{\mathcal{P}}_2{}''' \\ &- \bar{\mathcal{P}}_1{}'''\bar{\mathcal{P}}_2 + 2(\bar{\mathcal{P}}_1{}''\bar{\mathcal{P}}_2{}' - \bar{\mathcal{P}}_1{}'\bar{\mathcal{P}}_2{}''')]dt, \\ V &= 2\partial\partial \ln iW, \end{aligned}$$

is a solution of Novikov-Veselov equation, which is rational in $z, \bar z, t$.

Taimanov and Tsarev considered the polynimial-type solutions for (6)

$$P_N(t,z) = z^N + \sigma_1 z^{N-1} + \sigma_2 z^{N-2} + \cdots + \sigma_{N-1} z + \sigma_N.$$

Then the flow (6) generates the σ-flow:

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Then the flow (6) generates the σ-flow:

$$\dot{\sigma}_k = (N-K+3)(N-k+2)(N-k+1)\sigma_{k-3}, k = 1,2,3\cdots N.$$

It can be seen that σ_1, σ_2 are conserved quantities. Indeed, $\sigma_1, \sigma_2, \cdots, \sigma_N$ are the elementary symmetric polynomials in the roots $q_1, q_2, \cdots, q_N$ of $P(z)$:

$$\begin{aligned} \sigma_1(\vec{q}) &= -\sum_{i=1}^{N} q_i, \qquad \sigma_2(\vec{q}) = \sum_{i<j} q_i q_j, \\ \sigma_3(\vec{q}) &= -\sum_{i<j<k} q_i q_j q_k, \cdots, \sigma_N(\vec{q}) = (-1)^N q_1 q_2 \cdots q_N. \end{aligned} \quad (7)$$

The integrable (even linear) evolution of $\vec{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ induces a dynamical system on the symmetric product $S^N C$ of the complex roots . We call such a dynamical system on $S^N C$ a σ-system.

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Remark:

From (7), we see that given two solutions $\mathcal{P}_1(t, z)$ and $\mathcal{P}_2(t, z)$, by a substitution of $e^{i\lambda_1}\mathcal{P}_1(t, z)$ and $e^{i\lambda_2}\mathcal{P}_2(t, z)$, where λ_1 and λ_2 are real-valued constants, into (7) we obtain a solution of the Novikov-Veselov solution. Therefore, to each pair of holomorphic solutions of (6), we can get an $(S^1 \times S^1)$-family of solutions to the Novikiv-Veselov equation.

Let's write $P_N(t,z)$ as

$$P_N(t,z) = (z - q_1(t))(z - q_2(t)) \cdots (z - q_N(t)).$$

Then from the equation (6), one gets the root dynamics

$$\dot{q}_j = -6 \sum_{m<n, \quad j \neq m,n}^{N} \frac{1}{(q_j - q_m)(q_j - q_n).} \tag{8}$$

For example, when N=3, we have

$$\begin{aligned} \dot{q}_1 &= -6 \frac{1}{(q_1 - q_2)(q_1 - q_3)} \\ \dot{q}_2 &= -6 \frac{1}{(q_2 - q_1)(q_2 - q_3)} \\ \dot{q}_3 &= -6 \frac{1}{(q_3 - q_1)(q_3 - q_2)} \end{aligned}$$

For N=4, one has

$$\begin{aligned}
\dot{q_1} &= -6[\frac{1}{(q_1-q_2)(q_1-q_3)}+\frac{1}{(q_1-q_3)(q_1-q_4)}+\frac{1}{(q_1-q_2)(q_1-q_4)}\\
\dot{q_2} &= -6[\frac{1}{(q_2-q_1)(q_2-q_3)}+\frac{1}{(q_2-q_3)(q_2-q_4)}+\frac{1}{(q_2-q_1)(q_2-q_4)}\\
\dot{q_3} &= -6[\frac{1}{(q_3-q_1)(q_3-q_2)}+\frac{1}{(q_3-q_1)(q_3-q_4)}+\frac{1}{(q_3-q_2)(q_3-q_4)}\\
\dot{q_4} &= -6[\frac{1}{(q_4-q_2)(q_4-q_3)}+\frac{1}{(q_4-q_1)(q_4-q_2)}+\frac{1}{(q_4-q_1)(q_1-q_3)}
\end{aligned}$$

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\dot{q_2} &= -6[\frac{1}{(q_2-q_1)(q_2-q_3)}+\frac{1}{(q_2-q_3)(q_2-q_4)}+\frac{1}{(q_2-q_1)(q_2-q_4)}\\
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\dot{q_4} &= -6[\frac{1}{(q_4-q_2)(q_4-q_3)}+\frac{1}{(q_4-q_1)(q_4-q_2)}+\frac{1}{(q_4-q_1)(q_1-q_3)}
\end{aligned}$$

We notice that since σ_1 and σ_2 are conserved quantities, one knows that

$$\sum_{i=1}^{N} q_i, \qquad \sum_{i=1}^{N} q_i^2$$

are conserved densities of (8)

The goal is to investigate the properties of the root dynamics (8):

- Initial Value Problem
- Lax Pair
- Asymptotic behavior

The generating function of the Gould-Hopper polynomials $P_N(t,z)$ is

$$e^{\lambda z+\lambda^3 t}=\sum_{N=0}^{\infty}P_N(t,z)\frac{\lambda^N}{N!}.$$

Indeed, the Gould-Hopper polynomials $P_N(t,z)$ has the operator representation [1962]

$$P_N(t,z)=e^{t\partial_z^3}z^N=[1+t\partial_z^3+\frac{t^2\partial_z^6}{2!}+\frac{t^3\partial_z^9}{3!}+\frac{t^4\partial_z^{12}}{4!}+\cdots]z^N.$$

We remark that in general the Gould-Hopper polynomials are defined by $P_N^{(m)}(t,z)=e^{t\partial_z^m}z^N$. Here we take $m=3$.

One notices that the Gould-Hopper polynomials $P_N(t,z)$ are characterized by (6) and $P_N(0,z)=z^N$. For example,

$$\begin{aligned}
P_0 &= 1, \quad P_1 = z, \quad P_2 = z^2, \quad P_3 = z^3+6t, \quad P_4 = z^4+24tz, \\
P_5 &= z^5+60tz^2, \quad P_6 = z^6+120tz^3+360t^2 \\
P_7 &= z^7+210tz^4+2520z, \quad P_8 = z^8+336tz^5+10080t^2z^2 \\
P_9 &= z^9+504tz^6+30240t^2z^3+60480t^3 \\
P_{10} &= z^{10}+720tz^7+75600t^2z^4+604800t^3z
\end{aligned}$$

Actually, we have

$$\begin{aligned}
P_N(t,z) &= N!\sum_{k=0}^{[N/3]}\frac{t^k z^{N-3k}}{k!(N-3k)!} \\
\frac{dP_N(t,z)}{dz} &= NP_{N-1}(t,z) \qquad (9)
\end{aligned}$$

From the operation calculus, one has

$$(z + 3t\partial_z^2)P_N(t,z) = P_{N+1}(t,z).$$

Hence we yield the recursive relation

$$P_{N+1}(t,z) = zP_N(t,z) + 3t(N-1)(N-2)P_{N-3}(t,z). \qquad (10)$$

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We notice that if we consider the equation (6) with the initial data of analytical function

$$P(0,z) = \sum_{N=0}^{\infty} \alpha_N z^N,$$

then the formal solution is

$$P(t,z) = e^{t\partial_z^3} \sum_{N=0}^{\infty} \alpha_N z^N = \sum_{N=0}^{\infty} \alpha_N P_N(t,z). \qquad (11)$$

Remark:
The successive operations of the operator $(z+3t\partial_z^2)$ on the solution (11) can help us construct more solutions of (6). For example, if $P(0,z)=\sin z$, then we have,

$$\begin{aligned} e^{t\partial_z^3}\sin z &= e^{t\partial_z^3}\sum_{N=0}^{\infty}\frac{(-1)^N}{(2N+1)!}z^{2N+1}=\sum_{N=0}^{\infty}\frac{(-1)^N}{(2N+1)!}P_{2N+1}(t,z) \\ &= \sin(z-t). \end{aligned}$$

The last equation uses the fact $e^{t\partial_z^3}e^{iz}=e^{iz+t}$. Hence

$$(z+3t\partial_z^2)^N\sin(z-t), \quad N=0,1,2,3,4,\cdots$$

are also solutions of (6).

- Initial Value Problem : The root dynamics of σ-flow can be solved by

$$\begin{aligned} H_N(t,z) &= (z-q_1(t))(z-q_2(t))\cdots(z-q_N(t)) \\ &= P_N(t,z)+C_1P_{N-1}(t,z)+\cdots+C_NP_0(t,z), \end{aligned}$$

where the constants $C_1, C_2, \cdots, C_{N-1}, C_N$ are determined by the initial values of $q_1(0), q_2(0), \cdots, q_N(0)$, that is,

$$\begin{aligned} C_1 &= -\sum_{i=1}^{N} q_i(0), \qquad C_2 = \sum_{i<j} q_i(0)q_j(0), \\ C_3 &= -\sum_{i<j<k} q_i(0)q_j(0)q_k(0), \quad \cdots, \\ C_N &= (-1)^N q_1(0)q_2(0)\cdots q_N(0). \end{aligned}$$

It can seen that the solutions $q_1(t), q_2(t), \cdots, q_N(t)$ can be obtained algebraically.

- Lax pair:
  Firstly, we study the root dynamics of the Gould-Hopper polynomials, which correspond to the initial values $q_1(0) = q_2(0) = \cdots = q_N(0) = 0$.
  Let's define the $N \times N$ matrix by

$$X(t) = \begin{cases} a_{i,i+1} = 1, & \text{if } i = 1, 2, 3, \cdots, N\ ; \\ a_{i,i-2} = -3t(i-1)(i-2), & \text{if } i = 3, 4 \cdots, N-1; \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

  Then from the recursive relation (10), one knows that

$$P_N(t, z) = det(zI_N - X(t)).$$

  For example, when $N = 3$,

$$X(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6t & 0 & 0 \end{pmatrix};$$

N=4,

$$X(t)=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6t & 0 & 0 & 1 \\ 0 & -18t & 0 & 0 \end{pmatrix};$$

N=5,

$$X(t)=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -6t & 0 & 0 & 1 & 0 \\ 0 & -18t & 0 & 0 & 1 \\ 0 & 0 & -36t & 0 & 0 \end{pmatrix}.$$

We can write $X(t)$ as

$$X(t) = R(t)QR^{-1}(t),$$

where $Q = diag(q_1(t), q_2(t), \cdots, q_N(t)$ and

$$R(t) = \begin{pmatrix} P_0(q_1,t) & P_0(q_2,t) & P_0(q_3,t) & \cdots & P_0(q_N,t) \\ P_1(q_1,t) & P_1(q_2,t) & P_1(q_3,t) & \cdots & P_1(q_N,t) \\ P_2(q_1,t) & P_2(q_2,t) & P_2(q_3,t) & \cdots & P_2(q_N,t) \\ \vdots & & & & \\ P_N(q_1,t) & P_N(q_2,t) & P_N(q_3,t) & \cdots & P_N(q_N,t) \end{pmatrix}. \quad (13)$$

For instance, when $N = 3$,

$$R(t) = \begin{pmatrix} 1 & 1 & 1 \\ q_1 & q_2 & q_3 \\ q_1^2 & q_2^2 & q_3^2 \end{pmatrix};$$

N=4,

$$R(t) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ q_1 & q_2 & q_3 & q_4 \\ q_1^2 & q_2^2 & q_3^2 & q_4^2 \\ q_1^3 + 6t & q_2^3 + 6t & q_3^3 + 6t & q_4^3 + 6t \end{pmatrix};$$

N=5,

$$R(t) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ q_1 & q_2 & q_3 & q_4 & q_5 \\ q_1^2 & q_2^2 & q_3^2 & q_4^2 & q_5^2 \\ q_1^3 + 6t & q_2^3 + 6t & q_3^3 + 6t & q_4^3 + 6t & q_5^3 + 6t \\ q_1^4 + 24t & q_2^4 + 24t & q_3^4 + 24t & q_4^4 + 24t & q_5^4 + 24t \end{pmatrix}.$$

From the initial value problem, we notice that the polynomials t^n can be replaced by the elementary symmetric polynomials of the roots $q_1, q_2, \cdots, q_N$. Hence one has $R(\vec{q})$. It can be seen that

$$\dot{X}(t) = RLR^{-1},$$

where

$$L = \dot{Q} + [M, Q], \quad M = R^{-1}\dot{R}.$$

For example, when N=3,

$$L(t)=\begin{pmatrix} \dot{q}_1 & \dot{q}_2\frac{q_2-q_3}{q_3-q_1} & \dot{q}_3\frac{q_3-q_2}{q_2-q_1} \\ \dot{q}_1\frac{q_1-q_3}{q_3-q_2} & \dot{q}_2 & \dot{q}_3\frac{q_3-q_1}{q_1-q_2} \\ \dot{q}_1\frac{q_1-q_2}{q_2-q_3} & \dot{q}_2\frac{q_2-q_1}{q_1-q_3} & \dot{q}_3 \end{pmatrix};$$

N=4,

$$L(t)=\begin{pmatrix} \dot{q}_1 & -\frac{\dot{q}_2(q_2-q_3)(q_2-q_4)+6}{(q_1-q_3)(q_1-q_4)} & -\frac{\dot{q}_3(q_3-q_2)(q_3-q_4)+6}{(q_1-q_2)(q_1-q_4)} & -\frac{\dot{q}_4(q_4-q_2)(q_4-q_3)+6}{(q_1-q_2)(q_1-q_3)} \\ -\frac{\dot{q}_1(q_1-q_3)(q_1-q_4)+6}{(q_2-q_3)(q_2-q_4)} & \dot{q}_2 & -\frac{\dot{q}_3(q_3-q_1)(q_3-q_4)+6}{(q_2-q_4)(q_2-q_1)} & -\frac{\dot{q}_4(q_4-q_1)(q_4-q_3)+6}{(q_2-q_3)(q_2-q_1)} \\ -\frac{\dot{q}_1(q_1-q_2)(q_1-q_4)+6}{(q_2-q_3)(q_4-q_3)} & -\frac{\dot{q}_2(q_2-q_4)(q_2-q_1)+6}{(q_4-q_3)(q_1-q_3)} & \dot{q}_3 & -\frac{\dot{q}_4(q_4-q_1)(q_4-q_2)+6}{(q_2-q_3)(q_1-q_3)} \\ -\frac{\dot{q}_1(q_1-q_2)(q_1-q_3)+6}{(q_4-q_3)(q_4-q_2)} & -\frac{\dot{q}_2(q_2-q_3)(q_2-q_1)+6}{(q_4-q_3)(q_4-q_1)} & -\frac{\dot{q}_3(q_3-q_1)(q_3-q_2)+6}{(q_4-q_1)(q_4-q_2)} & \dot{q}_4 \end{pmatrix}$$

Since

$$\frac{dX(t)}{dt} = \begin{cases} a_{i,i-2} = -3(i-1)(i-2), & \text{if } i = 3, 4 \cdots, N-1; \\ 0, & \text{otherwise,} \end{cases}$$

we know $\frac{dX(t)}{dt}$ is a nilpotent matrix and hence L is a nilpotent one, too. So

$$tr(L^r) = tr[\frac{dX(t)}{dt}]^r = 0, \quad r = 1, 2, 3, \ldots, ..$$

Actually, a simple calculation yields

$$L^{[\frac{N}{2}]+1} = 0, \quad N \geq 3.$$

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Now,

$$\frac{d^2X(t)}{dt^2} = 0$$

will imply the Lax equation

$$\frac{dL(t)}{dt} = [L, M]. \tag{14}$$

For $N = 3, 4, 5$, we see that, by Maple software, q_i satisfies the following Goldfish model [Calogero, 2001], a limiting case of the Ruijesenaars-Schneider system:

$$\ddot{q}_i = 2\sum_{j\neq i} \frac{\dot{q}_i\dot{q}_j}{q_i - q_j}.$$

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The reason is that the Gould-Hopper polynomials $P_N, N = 3, 4, 5$ are linear in t-variable. For $N = 6$, we have from the diagonal terms of the Lax equation (14)

$$\ddot{q}_i = 2\sum_{j\neq i}^{6} \frac{\dot{q}_i\dot{q}_j}{q_i - q_j} + \frac{\sum_{j=1}^{6}(\text{some quadratic terms of } \vec{q})q_j + 720}{\prod_{i\neq j}^{6}(q_i - q_j)}.$$

Remark:
For the Goldfish Model

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its initial value problem can be solved by the statement:
$z = q_i(t), i = 1, 2, \cdots, N$ are the N roots of the equation

$$\sum_{i=1}^{N}\frac{\dot{q}_i(0)}{z - q_i(0)} = \frac{1}{t}.$$

It can be seen that it is a polynomial in z with coefficients linear in t. Then the special choices of initial datum can get the solutions of the root dynamics (8) for the cases $N = 3, 4, 5$.

Secondly, we consider the general case. Let's define the 2D Appell polynomials $\mathbb{G}_n(z,t)$ by means of the generating function [G.Bretti and P.Ricci, 2004]:

$$A(\lambda)e^{\lambda z+\lambda^3 t}=\sum_{n=0}^{\infty}\mathbb{G}_n(z,t)\frac{\lambda^n}{n!}, \tag{15}$$

where

$$A(\lambda)=\sum_{0}^{N}\frac{\mathbf{\Gamma}_k}{k!}\lambda^k,$$

$\mathbf{\Gamma}_k's$ being constants and $\mathbf{\Gamma}_0=1$. Then one has the following formula, noting that $\binom{N}{h}=\binom{N}{N-h}$,

$$\mathbb{G}_N=\sum_{h=0}^{N}\binom{N}{N-h}\mathbf{\Gamma}_{N-h}P_h(z,t). \tag{16}$$

It's easy to see that the polynomials $\mathbb{G}_n(z,t)$ also satisfy the linear equation (6).

Then we have

$$\mathbf{\Gamma}_{N-h} = \frac{C_{N-h}}{\binom{N}{N-h}}. \tag{17}$$

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Now, it's suitable to introduce the coefficients of the Taylor expansion

$$\frac{A'(\lambda)}{A(\lambda)} = \sum_{n=0}^{\infty} \alpha_n \frac{\lambda^n}{n!}.$$

It can be seen that the coefficients α_n can be expressed by $\mathbf{\Gamma}_0, \mathbf{\Gamma}_1, \cdots, \mathbf{\Gamma}_{n+1}$ (or the initial values (17)). For examples,

$$\begin{aligned}
\alpha_0 &= \mathbf{\Gamma}_1, \quad \alpha_1 = \mathbf{\Gamma}_2 - \mathbf{\Gamma}_1^2, \quad \alpha_2 = \mathbf{\Gamma}_3 - 3\mathbf{\Gamma}_1\mathbf{\Gamma}_2 + 2\mathbf{\Gamma}_1^3, \\
\alpha_3 &= \mathbf{\Gamma}_4 + 12\mathbf{\Gamma}_1^2\mathbf{\Gamma}_2 - 4\mathbf{\Gamma}_1\mathbf{\Gamma}_3 - 3\mathbf{\Gamma}_2^2 - 6\mathbf{\Gamma}_1^4, \\
&\vdots
\end{aligned}$$

The recurrence relation for the 2D Appell polynomial $\mathbb{G}_N(z,t)$ can be written as follows:

$$\begin{aligned} \mathbb{G}_0(z,t) &= 1 \\ \mathbb{G}_N(z,t) &= (z+\alpha_0)\mathbb{G}_{N-1}(z,t) + 3t(N-1)(N-2)\mathbb{G}_{N-3}(z,t) \\ &+ \sum_{k=0}^{N-2} \binom{N-1}{k} \alpha_{N-k-1}\mathbb{G}_k(z,t). \end{aligned} \tag{18}$$

A simple calculation can yield

$$\begin{aligned} \mathbb{G}_0(z,t) &= 1, \quad \mathbb{G}_1(z,t) = z+\alpha_0, \quad \mathbb{G}_2(z,t) = (z+\alpha_0)^2 + \alpha_1 \\ \mathbb{G}_3(z,t) &= (z+\alpha_0)^3 + 3\alpha_1(z+\alpha_0) + \alpha_2 + 6t \\ \mathbb{G}_4(z,t) &= (z+\alpha_0)^4 + 6\alpha_1(z+\alpha_0)^2 + (4\alpha_2+24t)(z+\alpha_0) + \alpha_3 \\ &+ 3\alpha_1^2. \end{aligned}$$

When $A(\lambda) = 1$, this recursive relation becomes (10). Hence the relation (18) is a generalization of (10) for arbitrary initial data.

The matrix corresponding to (12) can be constructed as follows:

$$X(t)=\begin{cases} a_{i,i+1}=1, & \text{if } i=1,2,3,\cdots,N-1\ ;\\ a_{i,i}=-\alpha_0, & \text{if } i=1,2,3,\cdots,N\ ;\\ a_{i,i-1}=-\binom{i-1}{i-2}\alpha_1, & \text{if } i=2,3,4,\cdots,N\ ;\\ a_{i,i-2}=-\binom{i-1}{i-3}(6t+\alpha_2), & \text{if } i=3,4,5,\cdots,N\ ;\\ a_{i,i-3}=-\binom{i-1}{i-4}\alpha_3, & \text{if } i=4,5,6,\cdots,N\ ;\\ \vdots & \\ a_{i,i-k}=-\binom{i-1}{i-k-1}\alpha_k, & \text{if } i=k+1,k+2,k+3,\cdots,N\ ;\\ \vdots & \\ a_{N,1}=-\alpha_{N-1} & \end{cases}$$

Similarly, one has

$$\mathbb{G}_N(z,t)=det(zI_N-X(t)).$$

For instance, when $N = 5$, we get

$$X(t) = \begin{pmatrix} -\alpha_0 & 1 & 0 & 0 & 0 \\ -\alpha_1 & -\alpha_0 & 1 & 0 & 0 \\ -(6t+\alpha_2) & -2\alpha_1 & -\alpha_0 & 1 & 0 \\ -\alpha_3 & -(18t+3\alpha_2) & -3\alpha_1 & -\alpha_0 & 1 \\ -\alpha_4 & -4\alpha_3 & -(36t+6\alpha_2) & -4\alpha_1 & -\alpha_0 \end{pmatrix}.$$

Also, one can write $X(t)$ as

$$X(t) = R(t)QR^{-1}(t),$$

where $Q = diag(q_1(t), q_2(t), \cdots, q_N(t))$ and $R(t)$ is defined as (13) with $P_m(q_i, t)$ being replaced by $\mathbb{G}_m(q_i, t)$. Then one follows the previous procedures and finally can get the Lax equation (14) for general case. Therefore the root dynamics (8) is Lax-integrable.

We notice here that for $N = 3, 4, 5$ the root dynamics of $\mathbb{G}_N$ also satisfies the Gold-fish model .

- Asymptotic behavior

It is known that the Gould-Hopper polynomial $P_N(t.z)$ has the scaling property:

$$P_N(t,z) = t^{\frac{N}{3}} \hat{P}_N(\frac{z}{t^{1/3}}), \qquad (19)$$

where $\hat{P}_N(\eta)$ is the so-called generalized Hermite polynomial (or Appell polynomial) in $\eta = \frac{z}{t^{1/3}}$. For example,

$$\begin{aligned} P_8(t,z) &= z^8 + 336tz^5 + 10080t^2 = t^{\frac{8}{3}}[\eta^8 + 336\eta^5 + 10080\eta^2] \\ &= t^{\frac{8}{3}} \hat{P}_8(\eta). \end{aligned}$$

Then the k-th zero $\lambda_N^{(k)}$ (constant)of $\hat{P}_N(\eta)$ determines the dynamics of the root q_k, i.e.,

$$q_k(t) = \lambda_N^{(k)} t^{1/3}.$$

Since $\hat{P}_N(\xi\lambda_N^{(k)}) = 0, \xi^3 = 1,$ one knows that the roots q_k are located on the circles in the plane with time dependent radius or fixed at the origin.

Since $\hat{P}_N(\xi\lambda_N^{(k)}) = 0, \xi^3 = 1,$ one knows that the roots q_k are located on the circles in the plane with time dependent radius or fixed at the origin. Finally, from the Initial value Problem, we know that when $t \to \infty$ and $z \to \infty$ such that $|z|^3/t \to$ constant, $P_N(t, z)$ plays the dominant role. Hence one yields

$$q_k(t) \to \lambda_N^{(k)} t^{1/3}.$$

Consequently, the roots asymptotically will follow diagonal lines.

In this section, we establish smooth rational solutions for all time by the extended Moutard transformation (7) and Gould -Hopper polynomials.

Example 1

Let $\mathcal{P}_1 = z^2 + z + 1$ and $\mathcal{P}_2 = -iz^2 - 2iz$. Then a simple calculation can yield the imaginary part of W in (7)

$$M(x, y, t) = (x^2 + y^2)^2 + \frac{8}{3}x^3 + 4xy^2 + 4x^2 + 4x + 4t + 100.$$

We can see that

$$M(x, y, 0) \approx 4x + 100 \quad \text{near} \quad (0, 0)$$

and

$$M(x, y, 0) \approx (x^2 + y^2)^2 \quad \text{near} \quad r = \sqrt{x^2 + y^2} = \infty.$$

It can be verified that $M(x, y, 0)$ is positive for all $\mathbb{R}^2$. Also, $M(x, y, t)$ is positive for all $\mathbb{R}^2$ at any fixed time t . Then the solution U of the Novikov-Veselov equation (1) is

where

$$\begin{aligned} M_1 &= -12\,(294 + 600\,x^2 + 588\,y^2 + 8\,x^3 + 888\,x + 12\,t + 3\,x^4 \\ &- 6\,x^2y^2 - 2\,x^3y^2 + 24\,x^2t - 3\,y^4x + 24\,y^2t + 36\,xt - 3\,y^4 + x^5) \end{aligned}$$

and

$$\begin{aligned} M_2 &= (3\,x^4 + 6\,x^2y^2 + 3\,y^4 + 8\,x^3 + 12\,xy^2 + 12\,x^2 \\ &+ 12\,x + 12\,t + 300)^2 \end{aligned}$$

At fixed time t , one knows U decays like $\frac{1}{r^3}$ for $r \to \infty$. Also, U tends asymptotically to zero at the rate $\frac{1}{t}$ at any fixed point (x, y) when $t \to \infty$.

Example 2

Let $\mathcal{P}_1 = (z^3 + 6t) + 2iz$ and $\mathcal{P}_2 = -i(z^3 + 6t) + z$. Then a simple calculation can yield the imaginary part of W in (7)

$$\begin{aligned} f(x, y, t) &= (x^2 + y^2)^3 + 4x^3 y + 8xy^3 + 2(x^2 + y^2) \\ &+ 6t(2x^3 - 6xy^2 - y) + 36t^2 + 6000. \end{aligned}$$

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Then we can see that

$$f(x,y,0) \approx 2(x^2+y^2) + 6000 \quad \text{near} \quad (0,0)$$

and

$$f(x,y,0) \approx (x^2+y^2)^3 \quad \text{near} \quad r = \sqrt{x^2+y^2} = \infty.$$

It can be verified that $f(x,y,0)$ is positive for all $\mathbb{R}^2$.

Also, letting $f(x,y,t)$ be equal to zero, one has

$$\begin{aligned} t &= (1/2)xy^2 + (1/12)y - (1/6)x^3 \\ &\pm (1/12)\sqrt{24x^2y^4 - 20xy^3 - 36x^4y^2 - 7y^2 - 20x^3y - 4y^6 - 8x^2 - 23996}. \end{aligned}$$

A simple calculation shows that the equation inside the square root is negative for all $\mathbb{R}^2$. Hence $f(x,y,t)$ is positive for all $\mathbb{R}^2$ at any fixed time t. Then a solution U of the Novikov-Veselov equation (1) is

$$U = \frac{F_1}{F_2},$$

where

$$
\begin{aligned}
F_1 &= -\frac{1}{2}[24\,x^7y + 16\,x^6 + 24\,x^5y^3 + \left(432\,ty + 216000 + 48\,y^2\right)x^4 \\
&+ \left(-24\,y^5 - 144\,t + 48\,y\right)x^3 + \left(432000\,y^2 + 864\,ty^3 - 96\,y^4\right)x^2 \\
&+ \left(-24\,y^7 - 48\,y^3 + 432\,ty^2 + \left(432000 + 1728\,t^2\right)y\right)x + 48000 \\
&+ 432\,y^5t - 32\,y^6 + 216000\,y^4 + 252\,t^2]
\end{aligned}
$$

and

$$
\begin{aligned}
F_2 &= (x^6 + 3\,x^4y^2 + 3\,x^2y^4 + y^6 + 4\,x^3y + 8\,xy^3 + 2\,x^2 + 2\,y^2 \\
&+ 12\,tx^3 - 36\,txy^2 - 6\,ty + 36\,t^2 + 6000)^2
\end{aligned}
$$

At fixed time t , one knows U decays like $\frac{1}{r^4}$ for $r \to \infty$. Also, U tends asymptotically to zero at the rate $\frac{1}{t^2}$ at any fixed point (x, y) when $t \to \infty$.

Example 3
Let $\mathcal{P}_1 = (z^5 + 60tz^2) + 2iz$ and $\mathcal{P}_2 = -i(z^5 + 60tz^2) + z$. Then the imaginary part of W in (7) is

$$
\begin{aligned}
& h(x,y,t) \\
= \; & \left((x^2+y^2)^5 - \frac{10}{3}\, x^5 y + \frac{20}{3}\, x^3 y^3 \right. \\
+ \; & (x^2+y^2)\left(2 + 12\, x^3 y - 12\, x y^3\right) \\
+ \; & 120\, t\, (x^2+y^2)\left(x^5 - 3\, y^4 x - 2\, x^3 y^2\right) + 3600\, t^2 \left(x^2+y^2\right)^2 \\
+ \; & 20\, t\left(11\, y^3 + 3\, x^2 y\right) + 120\, t^2 + 1000\right).
\end{aligned}
$$

Similarly, $h(x,y,t)$ is positive for all $\mathbb{R}^2$ at fixed time t. Then the solution U is

$$
U = \frac{H_1}{H_2},
$$

where

$$
\begin{aligned}
& H_1(x,y,t) \\
= \quad & -\frac{1}{2}[360\,xy^{13} + \left(360\,x^3 + 14400\,t\right)y^{11} - 144\,y^{10} + \left(14400\,x^2t - 1800\,x^5\right)y^9 \\
+ \quad & \left(-720\,x^2 + 900000 + 108000\,t^2\right)y^8 + \left(237600\,tx^4 - 777600\,t^2x - 3600\,x^7\right)y^7 \\
+ \quad & \left(3840\,x^4 + \left(3600000 + 432000\,t^2\right)x^2 + 83520\,tx\right)y^6 + \left(165600\,tx^6 + 720\,x\right. \\
- \quad & \left. 10368000\,t^3 - 1800\,x^9 - 345600\,t^2x^3\right)y^5 + \left(-1120\,x^6 + (5400000\right. \\
+ \quad & \left. 648000\,t^2\right)x^4 + 12000\,tx^3 + \left(-129600000\,t - 15552000\,t^3\right)x - 810000\,t^2\Big)y^4 \\
+ \quad & \left(360\,x^{11} - 50400\,x^8t + 950400\,x^5t^2 - 2160\,x^3 + 20736000\,t^3x^2 + \left(-216000\,t^2\right.\right. \\
- \quad & \left.\left. 1800000\right)x - 5760\,t\right)y^3 + \left(2380\,x^8 + \left(3600000 + 432000\,t^2\right)x^6 - 25200\,tx^5\right. \\
+ \quad & \left.\left(-10368000\,t^3 - 86400000\,t\right)x^3 + 453600\,x^2t^2 + 62208000\,t^4 + 518400000\,t^2\right)y^2 \\
+ \quad & \left(360\,x^{13} + 21600\,x^{10}t + 518400\,x^7t^2 + 720\,x^5 + 31104000\,t^3x^4 + (1920000\right. \\
+ \quad & \left. 230400\,t^2\right)x^3 + 17280\,x^2t + 1555200\,t^3 + 12960000\,t\Big)y + 476\,x^{10} \\
+ \quad & \left(900000 + 108000\,t^2\right)x^8 + 25200\,x^7t + \left(43200000\,t + 5184000\,t^3\right)x^5 \\
+ \quad & 226800\,x^4t^2 + \left(62208000\,t^4 + 518400000\,t^2\right)x^2 + 72000 + 8640\,t^2\Big)]
\end{aligned}
$$

and

$$
\begin{aligned}
& H_2(x,y,t) \\
= \quad & (3\,x^{10} + 15\,x^8y^2 + 30\,x^6y^4 + 30\,y^6x^4 + 15\,y^8x^2 \\
+ \quad & 3\,y^{10} + 26\,x^5y + 20\,x^3y^3 \\
+ \quad & 6\,x^2 + 6\,y^2 - 36\,xy^5 + 360\,x^7t - 1800\,tx^3y^4 - 360\,tx^5y^2 - 1080\,txy^6 \\
+ \quad & 10800\,x^4t^2 + 21600\,t^2x^2y^2 + 10800\,t^2y^4 + 660\,ty^3 + 180\,tx^2y \\
+ \quad & 360\,t^2 + 3000)^2
\end{aligned}
$$

In this case, at fixed time, U decays like $\frac{1}{r^6}$ for $r \to \infty$; moreover, we see that

$$
U \to \frac{-240(x^2+y^2)}{[30(x^2+y^2)^2+1]^2} = \frac{-240 z\bar{z}}{[30z^2\bar{z}^2+1]^2} \quad as \quad t \to \infty \qquad (20)
$$

at fixed point (x,y), which is a stationary solution of (1) for

$$
V = \frac{3600\,\bar{z}^4 z^2 - 120\,\bar{z}^2}{(30\,z^2\bar{z}^2+1)^2}.
$$

We notice that if we define

$$u(z,\bar{z}) = \ln U \quad and \quad V(z,\bar{z}) = \frac{U_{zz}}{3U},$$

then the stationary equation of the Novikov-Veselov equation (1) will become the Tzitzeica equation[B.C. Konopelchenko and U. Pinkall., 1998]

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$$u_{z\bar{z}} = e^{u} + \epsilon e^{-2u}, \tag{21}$$

where ϵ is an arbitrary constant. It can be verified that (20) satisfies the $\epsilon = 0$ case, i.e., the Liouville equation, whose real solutions are given by

$$\ln \frac{-2\kappa|\frac{dS}{dz}|^2}{(1+\kappa|S|^2)^2}, \tag{22}$$

where $S(z)$ is a locally univalent meromorphic function in some domain and κ is a constant. For the solution corresponding to (20), one knows that $S(z) = z^2$ and $\kappa = 30$.

For general case, let's choose

$$\mathcal{P}_1 = P_N(t,z) + 2iz \quad and \quad \mathcal{P}_2 = -iP_N(t,z) + z.$$

We can expect the imaginary part of W in (7) is positive for all $\mathbb{R}^2$ at any fixed time if we choose an appropriate constant. And the solution $U(x,y,t)$ at any fixed time decays like $\frac{1}{r^{N+1}}$ for N $\geq$ 2. When $t \to \infty$ and then one obtains the solutions (22) of the Liouville equation or the ones of the Tzitzeica equation (21).

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Remark: For each potential $U(x, y, t)$ there exist infinitely many wave functions, which can be constructed by the Pfaffian of linear combinations of the Gould-Hopper polynomials

In summary, one investigates

- The Gould-Hopper polynomials
- The σ-flow and the Lax pair
- The Gold-Fish model
- The smooth rational solutions of NV equation
- The solution of the Liouville equation (or the Tzitzeica equation)

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Thanks for your attention.