Cartan matrices and integrable lattice Toda field equations

Ismagil Habibullin

Ufa Institute of Mathematics, Ufa, Russia

Moscow 2011

Outline

- Algorithm of discretization of integrable models via integrals and symmetries
- Discretization of generalized 2D Toda lattices, corresponding to the Cartan matrices
- Toda lattices in discrete space-time and integrable cutting off conditions for the Hirota equation

Algorithm of discretization of Darboux integrable models via integrals

Begin with a simple observation. Liuoville Eq.:

$$u_{xy} = e^u$$

admits integrals $W = u_{xx} - 0.5u_x^2$, $\bar{W} = u_{yy} - 0.5u_y^2$ since $D_y W = 0$ and $D_x \bar{W} = 0$.

Differential-difference version of Liuoville Eq.

$$u_{1x} = u_x + Ce^{0.5(u+u_1)}$$

Integrals $W = u_{xx} - 0.5u_x^2$, and $F = e^{(u_1-u)/2} + e^{(u_1-u_2)/2}$, Since $DW = W$, and $D_x F = 0$. Here $u = u(x, y, n)$ and D is the shift operator $Du = u_1 = u(x, y, n+1)$. These two equations have a common integral W.

Differential-difference version of Liuoville Eq.

$$u_{1x} = u_x + Ce^{0.5(u+u_1)}$$

Integrals $W = u_{xx} - 0.5u_x^2$, and $F = e^{(u_1-u)/2} + e^{(u_1-u_2)/2}$, Since $D_n W = W$, and $D_x F = 0$. Here $u = u(x, n)$ and D_n is the shift operator $D_n u = u_1 = u(x, n+1)$.

Further discretization

$$e^{-u_{1,1}-u_{0,0}} = C + e^{-u_{1,0}-u_{0,1}}$$

where $u = u(n, m)$ and $u_{i,j} = u(n+i, m+j)$ admits integrals $F = e^{(u_{1,0}-u_{0,0})/2} + e^{(u_{1,0}-u_{2,0})/2}$, $I = e^{(u_{0,1}-u_{0,0})/2} + e^{(u_{0,1}-u_{0,2})/2}$. Because $D_n I = I$, $D_m F = F$, where D_m shifts the argument $D_m H(n, m) = H(n, m+1)$. These two equations have a common integral F.

Example

For the equation $u_{xy} = u_x u_y \left(\frac{1}{u-x} - \frac{1}{u-y} \right)$, (Goursat, 1899)
integrals are $W = \frac{u_{xx}}{u_x} - \frac{2u_x}{u-x} + \frac{1}{u-x}$, $\bar{W} = \frac{u_{yy}}{u_y} - \frac{2u_y}{u-y} + \frac{1}{u-y}$;

First discretization $u_{1x} = \frac{(u_1+L)(u_1-x)}{(u+L)(u-x)} u_x$ has integrals
$W = \frac{u_{xx}}{u_x} - \frac{2u_x}{u-x} + \frac{1}{u-x}$ and $F = \frac{(u_1-u)(u_2+L)}{(u_2-u)(u_1+L)}$,

Full discretization $u_{1,1} = \frac{L(u_{1,0}+u_{0,1}-u)+u_{1,0}u_{0,1}}{L+u}$
admits integrals

$F = \frac{(u_{1,0}-u_{0,0})(u_{2,0}+L)}{(u_{2,0}-u_{0,0})(u_{1,0}+L)}$ and $I = \frac{(u_{0,1}-u_{0,0})(u_{0,2}+L)}{(u_{0,2}-u_{0,0})(u_{0,1}+L)}$

I. Habibullin, N. Zheltukhina and A. Sakieva, *Discretization of hyperbolic type Darboux integrable equations preserving integrability*, J. Math. Phys. 52, 093507 (2011).

Algorithm of discretization via symmetries

The sine-Gordon equation $u_{x,y} = e^u + e^{-u}$ admits the following symmetries

$$u_t = u_{xxx} - 0.5{u_x}^3 \quad \text{and} \quad u_t = u_{yyy} - 0.5{u_y}^3$$

The semi-discrete chain $u_{1x} = -u_x + C_3 e^{0.5(u-u_1)}$ admits the symmetries

$$u_t = u_{xxx} - 0.5{u_x}^3 \quad \text{and} \quad u_t = e^{0.5(u_1 - u_{-1})}$$

Quad graph equation
$e^{\frac{1}{2}u_{1,1}} = P(m)e^{u_{0,0}+u_{1,0}-u_{0,1}} + (\alpha(m) + (-1)^n\beta(m))e^{\frac{1}{2}u_{0,0}}$ admits the symmetry

$$u_{0,0,t} = e^{0.5(u_{1,0}-u_{-1,0})}.$$

Habibullin, Zheltukhina, Sakieva, arxiv, 2011

Discretization of finite field Toda lattices, corresponding to the Cartan matrices of simple or affine Lie algebras

It is well known (see A. V. Mikhailov, M. A. Olshanetsky and A. M. Perelomov, Comm. Math. Phys. 79 (1981) 473.) that one can assign an integrable exponential type system of hyperbolic equations to each semi-simple or affine Lie algebra. Denote through $A = \{a_{ij}\}$ the Cartan matrix, corresponding to the Lie algebra, then the desired exponential system is written as follows

$$r^i_{x,y} = e^{\sum a_{ij} r^j}. \quad (1)$$

The first examples of (1) were discovered by Laplace and studied by Darboux, Goursat, Vessiot, etc. Modern theory of these systems is developed in works by many authors: Leznov, Shabat, Yamilov, Ibragimov, Bogoyavlensky, Drinfeld, Sokolov, Corrigan etc.

The system corresponding to the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{2}$$

canonically associated with the simple Lie algebra A_2 is of the form

$$r^1_{xy} = e^{2r^1 - r^2},$$
$$r^2_{xy} = e^{-r^1 + 2r^2}.$$

The system admits $y-$integrals

$$\begin{aligned} I^{(1)} &= w^{(1)} = r^1_{xx} + r^2_{xx} - (r^1_x)^2 + r^1_x r^2_x - (r^2_x)^2, \\ I^{(2)} &= w^{(2)} = r^1_{xxx} + r^1_x(r^2_{xx} - 2r^1_{xx}) + (r^1_x)^2 r^2_x - r^1_x (r^2_x)^2 \end{aligned}$$

Look for a system of differential-difference equations of the form

$$r^1_{1x} = r^1_x + f(r^1, r^1_1, r^2, r^2_1),$$
$$r^2_{1x} = r^2_x + g(r^1, r^1_1, r^2, r^2_1)$$

admitting the same integrals.

The answer is found in I. Habibullin, K. Zheltukhin and M. Yangubaeva, *Cartan matrices and integrable lattice Toda field equations*, J. Phys. A: Math. Theor. 44 (2011) 465202 (20pp).

the differential-difference chain

$$\begin{aligned} r^1_{1,x} - r^1_x &= e^{r^1 + r^1_1 - r^2_1}, \\ r^2_{1,x} - r^2_x &= e^{-r^1 + r^2 + r^2_1}. \end{aligned} \tag{3}$$

admits the same integrals (here subindices mean x-derivative and shift of the discrete argument):

$$\begin{aligned} I^{(1)} &= w^{(1)} = r^1_{xx} + r^2_{xx} - (r^1_x)^2 + r^1_x r^2_x - (r^2_x)^2, \\ I^{(2)} &= w^{(2)} = r^1_{xxx} + r^1_x(r^2_{xx} - 2r^1_{xx}) + (r^1_x)^2 r^2_x - r^1_x (r^2_x)^2 \end{aligned}$$

It is remarkable that this system admits also integrals in the second destination

$$F_1 = e^{-r^2 + r^2_1} + e^{-r^1 + r^1_1 + r^2_1 - r^2_2} + e^{r^1_1 - r^1_2},$$
$$F_2 = e^{-r^1 + r^1_1} + e^{r^1_1 - r^1_2 - r^2_1 + r^2_2} + e^{r^2_2 - r^2_3}.$$

Now the problem is to find a system of fully difference equations admitting the same pair of integrals.

The system is found in I.Habibullin, R.Garifullin *Affine Lie algebras and integrable Toda field equations on discrete space-time*, arXiv:1109.1689v2 [nlin.SI] 10 Oct 2011.

$$\begin{aligned} e^{-r^1_{1,1}+r^1_{1,0}+r^1_{0,1}-r^1_{0,0}}-1 &= e^{r^1_{0,1}+r^1_{1,0}-r^2_{1,0}}, \\ e^{-r^2_{1,1}+r^2_{1,0}+r^2_{0,1}-r^2_{0,0}}-1 &= e^{-r^1_{0,1}+r^2_{0,1}+r^2_{1,0}}, \end{aligned} \tag{4}$$

Rewrite the system in a compact form by setting

$$\Delta_{1,1}(r) := e^{-r_{1,1}+r_{1,0}+r_{0,1}-r_{0,0}}-1, \tag{5}$$

then the last system takes the form

$$\begin{aligned} \Delta_{1,1}(r^1) &= e^{r^1_{0,1}+r^1_{1,0}-r^2_{1,0}}, \\ \Delta_{1,1}(r^2) &= e^{-r^1_{0,1}+r^2_{0,1}+r^2_{1,0}}. \end{aligned} \tag{6}$$

Surprisingly it also admits a complete set of integrals

$$
\begin{aligned}
\Delta_{1,1}(r^1) &= e^{r^1_{0,1}+r^1_{1,0}-r^2_{1,0}}, \\
\Delta_{1,1}(r^2) &= e^{-r^1_{0,1}+r^2_{0,1}+r^2_{1,0}}.
\end{aligned} \tag{7}
$$

For this case (remind it corresponds to the algebra A_2) m-integrals are

$$
\begin{aligned}
F_1 &= e^{-r^2_{0,0}+r^2_{1,0}} + e^{-r^1_{0,0}+r^1_{1,0}+r^2_{1,0}-r^2_{2,0}} + e^{r^1_{1,0}-r^1_{2,0}}, \\
F_2 &= e^{-r^1_{0,0}+r^1_{1,0}} + e^{r^1_{1,0}-r^1_{2,0}-r^2_{1,0}+r^2_{2,0}} + e^{r^2_{2,0}-r^2_{3,0}}.
\end{aligned}
$$

and n-integrals are

$$
\begin{aligned}
I_1 &= e^{-r^1_{0,0}+r^1_{0,1}} + e^{-r^2_{0,0}+r^2_{0,1}+r^1_{0,1}-r^1_{0,2}} + e^{r^2_{0,1}-r^2_{0,2}}, \\
I_2 &= e^{-r^2_{0,0}+r^2_{0,1}} + e^{r^2_{0,1}-r^2_{0,2}-r^1_{0,1}+r^1_{0,2}} + e^{r^1_{0,2}-r^1_{0,3}}.
\end{aligned}
$$

System (7) admits higher symmetries. The lowest order symmetry is of the form:

$$
\begin{aligned}
\frac{dr^1_{0,0}}{dt} &= \frac{3e^{r^1_{0,0}-r^1_{1,0}}}{D_m^{-1}F_1} - 1, \\
\frac{dr^2_{0,0}}{dt} &= 1 - \frac{3e^{r^2_{0,0}-r^2_{-1,0}}}{D_m^{-1}F_1}.
\end{aligned}
$$

Discuss the result of discretization. We have systems corresponding to the matrix (algebra A_2)

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{aligned} r^1_{1,x} - r^1_x &= e^{r^1 + r^1_1 - r^2_1}, \\ r^2_{1,x} - r^2_x &= e^{-r^1 + r^2 + r^2_1}. \end{aligned}$$

$$\begin{aligned} \Delta_{1,1}(r^1) &= e^{r^1_{0,1} + r^1_{1,0} - r^2_{1,0}}, \\ \Delta_{1,1}(r^2) &= e^{-r^1_{0,1} + r^2_{0,1} + r^2_{1,0}}. \end{aligned}$$

Systems corresponding to the matrix (algebra B_2)

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

are

$$\begin{aligned} r^1_{1,x} - r^1_x &= e^{r^1 + r^1_1 - r^2_1}, \\ r^2_{1,x} - r^2_x &= e^{-2r^1 + r^2 + r^2_1}. \end{aligned}$$

$$\begin{aligned} \Delta_{1,1}(r^1) &= e^{r^1_{0,1} + r^1_{1,0} - r^2_{1,0}}, \\ \Delta_{1,1}(r^2) &= e^{-2r^1_{0,1} + r^2_{0,1} + r^2_{1,0}}. \end{aligned}$$

They admit complete set of integrals

Systems corresponding to the matrix (exceptional algebra G_2)

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

are

$$\begin{aligned} r^1_{1,x} - r^1_x &= e^{r^1 + r^1_1 - r^2_1}, \\ r^2_{1,x} - r^2_x &= e^{-3r^1 + r^2 + r^2_1}. \end{aligned}$$

$$\begin{aligned} \Delta_{1,1}(r^1) &= e^{r^1_{0,1} + r^1_{1,0} - r^2_{1,0}}, \\ \Delta_{1,1}(r^2) &= e^{-3r^1_{0,1} + r^2_{0,1} + r^2_{1,0}}. \end{aligned}$$

Each of them admits a complete set of integrals

In general we define the map assigning discrete exponential systems to any $N \times N$ matrix A with constant entries $a_{i,j}$.
Differential-difference system

$$v^i_{1,x} - v^i_x = e^{\sum_{j=1}^{j=i-1} a_{i,j} v^j + \sum_{j=i+1}^{j=N} a_{i,j} v^j_1 + \frac{1}{2} a_{i,i}(v^i + v^i_1)}, \quad i = 1, 2, ..., N$$

and difference-difference system

$$e^{-u^i_{1,1} + u^i_{1,0} + u^i_{0,1} - u^i_{0,0}} - 1 = e^{\sum_{j=1}^{j=i-1} a_{i,j} u^j_{0,1} + \sum_{j=i+1}^{j=N} a_{i,j} u^j_{1,0} + \frac{1}{2} a_{i,i}(u^i_{0,1} + u^i_{1,0})}$$

The subindex indicates shifts of the arguments n, m as follows $u^j_{i,k} := u^j(n+i, m+k)$.
Conjecture. a) If A is the Cartan matrix of a semi-simple Lie algebra then the system is Darboux integrable.
b) If A is the Cartan matrix of an affine Lie algebra then – S-integrable.

Lax pairs for the Toda chains with discrete space-time and for the finite field reductions of the Hirota equation

Finite Toda lattices in discrete space-time are closely connected with the integrable cutting off constraints for the Hirota equation

$$T_{0,0}T_{1,1} - T_{0,1}T_{1,0} = T^{1}_{0,1}T^{-1}_{1,0},$$

where $T = T(n, m, k)$ and $T^{s}_{i,j} = T(n+i, m+j, k+s)$. It admits Lax pair

$$\psi_{1,0} = -\psi^{1}_{0,0} + \frac{T_{0,0}T^{1}_{1,0}}{T_{1,0}T^{1}_{0,0}}\psi_{0,0},$$

$$\psi_{0,1} = \psi_{0,0} + \frac{T^{1}_{0,1}T^{-1}_{0,0}}{T_{0,1}T_{0,0}}\psi^{-1}_{0,0}.$$

By imposing $T(n,m,0)=1$ for all m,n and $T(n,m,N+1)=1$ for all m,n one finds a finite reduction of the system which corresponds to the discrete Toda system known to be integrable with the Cartan matrix of the simple Lie algebra A_N:

$$A=\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

Indeed by setting $T=e^{-r}$ one obtains R. S. Ward, Discrete Toda Field Equations, Phys. Lett. A, 199, 45 (1995).

$$\begin{aligned} \Delta_{1,1}(r^1) &= e^{r^1_{0,1}+r^1_{1,0}-r^2_{1,0}}, \\ \Delta_{1,1}(r^2) &= e^{-r^1_{0,1}+r^2_{0,1}+r^2_{0,1}-r^3_{1,0}}, \\ \dots\dots\dots\dots & \dots\dots \dots\dots\dots\dots\dots\dots\dots\dots \\ \Delta_{1,1}(r^{N-1}) &= e^{-r^{N-2}_{0,1}+r^{N-1}_{0,1}+r^{N-1}_{0,1}-r^N_{0,1}}, \\ \Delta_{1,1}(r^N) &= e^{-r^{N-1}_{0,1}+r^N_{0,1}+r^N_{0,1}}, \end{aligned}$$

Remind that $\Delta_{1,1}(r):=e^{-r_{1,1}+r_{1,0}+r_{0,1}-r_{0,0}}-1$.

By imposing $T(n+1,m,0)=T(n,m+1,2)$ and $T(n,m,k)=1$ for all $k\geq N+1$ we obtain a finite reduction corresponding to the simple Lie algebra of series C_N:

$$
\begin{array}{rcl}
\Delta_{1,1}(r^1) & = & e^{r^1_{0,1}+r^1_{1,0}-2r^2_{1,0}}, \\
\Delta_{1,1}(r^2) & = & e^{-r^1_{0,1}+r^2_{0,1}+r^2_{0,1}-r^3_{1,0}}, \\
\ldots\ldots\ldots\ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\Delta_{1,1}(r^{N-1}) & = & e^{-r^{N-2}_{0,1}+r^{N-1}_{0,1}+r^{N-1}_{0,1}-r^{N}_{0,1}}, \\
\Delta_{1,1}(r^N) & = & e^{-r^{N-1}_{0,1}+r^{N}_{0,1}+r^{N}_{0,1}},
\end{array}
$$

found in some different form in I. T. Habibullin, C-Series discrete chains, Theoretical and Mathematical Physics, 2006, 146:2, 170182

The examples above all are related to the simple Lie algebras. Consider now systems connected with affine algebras. By imposing $T(n+1,m,0) = T(n,m+1,2)$ and $T(n,m+1,N+1) = T(n+1,m,N-1)$ we obtain a system corresponding to the affine Lie algebra of series $D_N^{(2)}$ if $N \geq 3$:

$$\begin{array}{rcl} \Delta_{1,1}(r^1) & = & e^{r^1_{0,1}+r^1_{1,0}-2r^2_{1,0}}, \\ \Delta_{1,1}(r^2) & = & e^{-r^1_{0,1}+r^2_{0,1}+r^2_{0,1}-r^3_{1,0}}, \\ \ldots\ldots\ldots\ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Delta_{1,1}(r^{N-1}) & = & e^{-r^{N-2}_{0,1}+r^{N-1}_{0,1}+r^{N-1}_{0,1}-r^N_{0,1}}, \\ \Delta_{1,1}(r^N) & = & e^{-2r^{N-1}_{0,1}+r^N_{0,1}+r^N_{0,1}}, \end{array}$$

This system admits Lax representation. It is S-integrable for any N. For $N=2$ it corresponds to the affine algebra $A_1^{(1)}$

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

The system is of the form

$$T_{0,0}T_{1,1} - T_{1,0}T_{0,1} = (T^1_{0,1})^2,$$
$$T^1_{0,0}T^1_{1,1} - T^1_{1,0}T^1_{0,1} = (T_{1,0})^2.$$

or in terms of the variables $T = e^{-r}$

$$\Delta_{1,1}(r^1) = e^{r^1_{0,1}+r^1_{1,0}-2r^2_{1,0}},$$
$$\Delta_{1,1}(r^2) = e^{-2r^1_{0,1}+r^2_{0,1}+r^2_{1,0}}.$$

$$P_{-1,0} = \begin{pmatrix} \frac{T_{-1,0}T^1_{-1,1}}{T_{0,0}T^1_{-2,1}} & 0 & 0 \\ 0 & \frac{T_{-1,0}T^1_{-1,0}}{T_{0,0}T^1_{-2,0}} & 0 \\ 0 & 1 & 0 \end{pmatrix} P + \begin{pmatrix} \mu\frac{T_{-1,0}T^1_{-1,1}}{T_{0,0}T^1_{-2,1}} & 0 & 0 \\ \mu\frac{T_{-1,0}T^1_{-1,0}}{T_{0,0}T^1_{-2,0}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q,$$

$$P_{0,-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{T^1_{-1,0}T^1_{0,-1}}{T^1_{0,0}T^1_{-1,-1}} & 0 \\ 0 & 0 & 1 \end{pmatrix} P + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{(T_{0,-1})^2}{T^1_{0,0}T^1_{-1,-1}} \\ 0 & 0 & -\frac{T_{1,-1}T_{0,-1}}{T^1_{0,0}T^1_{0,-1}} \end{pmatrix} Q,$$

$$Q_{1,0} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{T_{0,0}T^1_{1,0}}{T^1_{0,0}T_{1,0}} & 0 \\ 0 & 0 & \frac{T^1_{1,0}T_{0,-1}}{T^1_{0,0}T_{1,-1}} \end{pmatrix} Q + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} P,$$

$$Q_{0,1} = \begin{pmatrix} \frac{T_{-1,0}T_{0,1}}{T_{0,0}T_{-1,1}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} Q + \begin{pmatrix} \frac{1}{\mu}\frac{(T^1_{-1,1})^2}{T_{0,0}T_{-1,1}} & 0 & 0 \\ \frac{1}{\mu}\frac{T^1_{0,1}T^1_{-1,1}}{T_{0,0}T_{0,1}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} P.$$

Our conjecture on integrability of the discrete systems

$$v^i_{1,x} - v^i_x = e^{\sum_{j=1}^{j=i-1} a_{i,j} v^j + \sum_{j=i+1}^{j=N} a_{i,j} v^j_1 + \frac{1}{2} a_{i,i}(v^i + v^i_1)}, \quad i = 1, 2, ..., N$$

and

$$e^{-u^i_{1,1}+u^i_{1,0}+u^i_{0,1}-u^i_{0,0}} - 1 = e^{\sum_{j=1}^{j=i-1} a_{i,j} u^j_{0,1} + \sum_{j=i+1}^{j=N} a_{i,j} u^j_{1,0} + \frac{1}{2} a_{i,i}(u^i_{0,1} + u^i_{1,0})}$$

for the Cartan matrix of any simple or affine Lie algebra is approved by finding Lax pairs in the following cases: A_N, C_N, B_N, $D_N^{(2)}$, A_1^1 and by finding integrals in the cases A_2, C_2, G_2.