Frobenius manifolds and 2+1 integrable systems

- The extended 2D Toda hierarchy -

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International Conference on
Geometrical Methods in Mathematical Physics

Moscow, 15 December 2011

- joint work with B. Dubrovin, L. Ph. Mertens.
- based on:

[1] G. C., B. Dubrovin, L. Ph. Mertens. *Math. Ann.* 349 (2011) 75–115. arXiv:0902.1245
[2] G. C., L. Ph. Mertens. arXiv:1109.5343

Frobenius manifolds and WDVV equations

- A Frobenius manifold $(M, \eta, \bullet, e, E)$ is given by
  - a commutative *associative* product $\bullet : TM \times TM \to TM$,
  - a *flat* invariant non-deg. bilinear form $\eta = <, >$ on TM ("metric"),
  - a *flat unit* vector field e, i.e. $\nabla e = 0$,
  - *integrability* of the structure constants i.e. $\exists$ a potential F,
  - *quasihomogeneity* w.r.t. a linear Euler vector field E: $\mathrm{Lie}_E \bullet = \bullet$, $\mathrm{Lie}_E \eta = (2-d)\eta$.

- In flat coordinates t_α the potential F satisfies the WDVV equations (Witten '90, Dijkgraaf, Verlinde, Verlinde '91)

$$\frac{\partial^3 F}{\partial t_\alpha \partial t_\beta \partial t_\gamma} \eta^{\gamma\delta} \frac{\partial^3 F}{\partial t_\delta \partial t_\rho \partial t_\sigma} = \frac{\partial^3 F}{\partial t_\sigma \partial t_\beta \partial t_\gamma} \eta^{\gamma\delta} \frac{\partial^3 F}{\partial t_\delta \partial t_\rho \partial t_\alpha}.$$

Bi-Hamiltonian structure on the loop space $\mathcal{L}M$

- To the Frobenius manifold M we associate a bi-Hamiltonian hierarchy of quasi-linear equations on the loop space

$$\mathcal{L}M = \{u : S^1 \to M\}.$$

- A flat metric g^{ij} on a manifold M induces a Poisson bracket of hydrodynamic type on $\mathcal{L}M$

$$\{F, G\} = \int_{S^1} \frac{\delta F}{\delta u^i} \left[g^{ij} \partial_x + \Gamma^{ij}_k u^k_x \right] \frac{\delta G}{\delta u^j}\, dx.$$

- Given a Frobenius manifold M, the metric η and the intersection form $g = i_E \bullet$ form a pencil of flat metrics, hence they induce compatible Poisson structures $\{,\}_1$, $\{,\}_2$, on the loop space $\mathcal{L}M$.

Deformed flat connection

- Define a connection $\tilde{\nabla}$ on $M \times \mathbb{C}^*$ by

$$\tilde{\nabla}_X Y = \nabla_X Y + \zeta X \bullet Y,$$
$$\tilde{\nabla}_{\frac{d}{d\zeta}} Y = \partial_\zeta Y + E \bullet Y - \frac{1}{\zeta}\mathcal{V} Y$$

  where $\mathcal{V} := \frac{2-d}{2} - \nabla E$ and $X, Y \in TM$, $\zeta \in \mathbb{C}^*$.
- The deformed connection $\tilde{\nabla}$ is flat.
- There exist deformed flat coordinates $\tilde{t}_\alpha(t, \zeta)$, i.e. such that

$$\tilde{\nabla} d\tilde{t}_\alpha = 0.$$

- The second deformed flatness equation is a singular matrix valued linear differential system on $\mathbb{C}$

$$\partial_\zeta \nabla \tilde{t}_\alpha = (\mathcal{U} + \frac{1}{\zeta}\mathcal{V})(\nabla \tilde{t}_\alpha).$$

- The monodromy (i.e. normal form at $\zeta = 0$ of the fundamental matrix) is independent of the point $t \in M$.
- Can choose deformed flat coordinates (Levelt basis) of the form

$$(\tilde{t}_1, \ldots, \tilde{t}_n) = (\theta_1, \ldots, \theta_n)\zeta^{\mathcal{V}}\zeta^{R}$$

where θ_α depend on $t \in M$ and analytically on $\zeta \in \mathbb{C}$ and the matrices μ, R are constant.

The principal hierarchy of M

- Expanding the deformed flat coordinates at $\zeta = 0$

$$\theta_\alpha(\zeta) = \sum_{p \geqslant 0} \theta_{\alpha,p}\zeta^p$$

one defines functions $\theta_{\alpha,p} : M \to \mathbb{C}$.

- The Hamiltonians $H_{\alpha,p} = \int_{S^1} \theta_{\hat{\alpha},p+1}\, dx$ define the commuting flows of the Principal hierarchy on $\mathcal{L}M$ by

$$\frac{\partial}{\partial t^{\alpha,p}} \cdot = \{\cdot, H_{\alpha,p}\}_1$$

where $\{,\}_1$ is the Poisson bracket of hydrodynamic type associated to the flat metric of M.

Examples

- $n = 1$: the principal hierarchy of the Frobenius manifold with potential $F = \frac{u^3}{6}$ coincides with the dispersionless limit $\epsilon \to 0$ of the KdV hierarchy:

$$u_t = u u_x - \epsilon u_{xxx} \quad \text{(KdV)}$$

- $n = 2$: the principal hierarchy of the charge $d = 1$ potential $F = \frac{v^2 u}{2} + e^u$ Frobenius mfd ($QH^*(\mathbb{C}P^1)$) is given by the dispersionless limit of the extended Toda hierarchy (GC, Dubrovin, Zhang 2004).

$2+1$ integrable systems

The 2D Toda hierarchy

- $2+1$ integrable equations e.g.

$$u_{yy} = (u_t - uu_x - \epsilon u_{xxx})_x \quad \text{(KP)}$$

$$u_{tt} - u_{yy} = e^{u(x+\epsilon)} - 2e^{u(x)} + e^{u(x-\epsilon)} \quad \text{(2D Toda)}$$

  are usually expressed as zero-curvature (or ZS) equations of hierarchies with infinite number of dependent variables.

- Lax representation for 2D Toda (Ueno, Takasaki '84)

$$L = \Lambda + u_0 + u_{-1}\Lambda^{-1} + \dots,$$

$$\bar{L} = \bar{u}_{-1}\Lambda^{-1} + \bar{u}_0 + \dots$$

  Lax equations:

$$L_{s_n} = [B_n, L], \quad L_{\bar{s}_n} = [\bar{B}_n, L]$$

$$\bar{L}_{s_n} = [B_n, \bar{L}], \quad \bar{L}_{\bar{s}_n} = [\bar{B}_n, \bar{L}]$$

  where $B_n = (L^n)_+$, $\bar{B}_n = (\bar{L}^n)_-$.

- ZS equations: $(B_n)_{s_m} - (B_m)_{s_n} + [B_n, B_m] = 0, \quad etc.$

- we consider the dispersionless $\epsilon \to 0$ limit, where

$$u_{tt} - u_{yy} = (e^u)_{xx} \quad \text{(d2D Toda)}$$

- Lax and ZS equations are obtained by the dispersive equations as follows:
  - substitute the Lax operators L, $\bar{L}$ with their symbols $\lambda(z)$, $\bar{\lambda}(z)$ by

$$\Lambda \to z$$

  - substitute the commutator $[,]$ with the Poisson bracket

$$\{f, g\} = z\frac{\partial f}{\partial z}\frac{\partial g}{\partial x} - z\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}.$$

Infinite-dimensional Frobenius manifolds

We are led to consider infinite-dimensional Frobenius manifolds, and correspondingly, solutions of WDVV depending on an infinite number of variables.

- WDVV solutions with an infinite number of primaries associated with d2D Toda tau functions describing conformal maps (Wiegmann, Zabrodin 2000; Boyarsky et al. Phys. Lett. B 2001)
- ∞-dim. Frobenius manifold for d2D Toda hierarchy (GC, Dubrovin, Mertens, Math. Ann. 2011, arXiv:0902.1245)
- Construction of deformed flat connection, principal hierarchy and extension of d2D Toda (GC, Mertens 2011, arXiv:1109.5343)
- ∞-dim.F.mfd. for dKP hierarchy (Raimondo 2010)
- ∞-dim.F.mfd. for 2-component B-KP hierarchy (Wu, Xu 2011).

A residue formula for a flat metric

on the space of analytic curves in $\mathbb{C}$

$$M_{S^1} = \left\{ w(z) \in \mathcal{H}(S^1) \text{ s.t. (1) and (2) are satisfied} \right\}$$

(1) $w'(z) \neq 0$ for $z \in S^1$,

(2) $\Gamma := w(S^1)$ is a non-selfintersecting positively oriented closed curve encircling the origin $w = 0$.

Proposition

The formula

$$\langle \partial_1, \partial_2 \rangle_{S^1} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{\partial_1 w(z)\ \partial_2 w(z)}{z^2 w'(z)}\, dz$$

defines a flat non-degenerate symmetric bilinear form on TM_{S^1}.

Construction of flat coordinates

- Riemann–Hilbert factorization of the inverse $z = z(w) : \Gamma \to S^1$

$$z(w) = f_0^{-1}(w) f_\infty(w) \quad \text{for} \quad w \in \Gamma$$

where $f_0(w)$ and $f_\infty(w)/w$ are holomorphic and non-vanishing inside/outside the curve Γ, and $f_\infty(w) = w + O(1), \quad |w| \to \infty$.

- The coefficients t_n of Taylor expansions of the logarithms

$$\log f_0(w) = -t_0 - t_1 w - t_2 w^2 - \dots, \quad |w| \to 0,$$
$$\log \frac{f_\infty(w)}{w} = \frac{t_{-1}}{w} + \frac{t_{-2}}{w^2} + \dots, \quad |w| \to \infty.$$

constitute a system of flat coordinates on M_{S^1}.

The manifold M

Let M be given by pairs

$$(\lambda(z), \bar{\lambda}(z)) \in z\mathcal{H}(D_\infty) \oplus \frac{1}{z}\mathcal{H}(D_0)$$

of holomorphic functions with

$$\lambda \sim z + \dots \text{ for } |z| \to \infty,$$

$$\bar{\lambda} \sim \frac{e^u}{z} + v + \dots \text{ for } |z| \to 0,$$

s.t.

(1) $w'(z) \neq 0$ for $|z| = 1$, where $w(z) := \lambda(z) + \bar{\lambda}(z) \in \mathcal{H}(S^1)$,

(2) the closed curve $\Gamma := w(S^1)$ is non-selfintersecting positively oriented encircling the origin $w = 0$,

(3) $\lambda(z)$, $\bar{\lambda}(z)$ are non-vanishing.

- The map

$$\begin{aligned} M &\to M_{S^1} \oplus \mathbb{C}^2 \\ (\lambda, \bar{\lambda}) &\mapsto (w(z), v, u) \end{aligned}$$

  is invertible by

$$\begin{aligned} \lambda(z) &= w_{\leqslant 0}(z) + z - e^u/z - v, \\ \bar{\lambda}(z) &= w_{\geqslant 1}(z) - z + e^u/z + v. \end{aligned}$$

- The metric on M is the direct sum of the flat metric on M_{S^1} and $ds^2 = dudv + dvdu$.

The product on T^*M

- Define basic 1-forms $d\lambda(p)$ and $d\bar{\lambda}(p)$ depending on $p \in S^1$ by

$$\langle d\lambda(p), \hat{\alpha}\rangle = \alpha(p), \quad \langle d\bar{\lambda}(p), \hat{\alpha}\rangle = \bar{\alpha}(p)$$

  where $\hat{\alpha} = (\alpha, \bar{\alpha}) \in TM = \mathcal{H}(D_\infty) \oplus \frac{1}{z}\mathcal{H}(D_0)$.
- Any 1-form $\hat{\omega} = (\omega(z), \bar{\omega}(z))$ can be represented as a linear combination of the 1-forms $d\lambda(p)$, $d\bar{\lambda}(p)$:

$$\hat{\omega} = \frac{1}{2\pi i}\oint_{|p|=1} \left(\omega(p)d\lambda(p) + \bar{\omega}(p)d\bar{\lambda}(p)\right).$$

Proposition

$$d\alpha(p) \bullet d\beta(q) = \frac{p\,q}{p-q}\left[\alpha'(p)\,d\beta(q) - \beta'(q)\,d\alpha(p)\right], \quad \alpha, \beta = \lambda, \bar{\lambda}$$

defines on T^*M a structure of a commutative associative algebra.

Idempotents and semisimplicity

- Define the family of 1-forms depending on the parameter $p \in S^1$

$$du(p) = \frac{\lambda'(p)}{\lambda'(p) + \bar{\lambda}'(p)} d\bar{\lambda}(p) - \frac{\bar{\lambda}'(p)}{\lambda'(p) + \bar{\lambda}'(p)} d\lambda(p)\,.$$

Proposition

The 1-forms $du(p)$ for $p, q \in S^1$ are idempotents of the Frobenius algebra on T^*M, in particular

$$du(p) \bullet du(q) = f(p)\delta(p-q)du(p), \quad f(p) = -p^2 \frac{\lambda'(p)\bar{\lambda}'(p)}{\lambda'(p) + \bar{\lambda}'(p)}.$$

- the Frobenius manifold M is semisimple.

The Frobenius manifold

Theorem

$(M, <, >, \bullet, e, E)$ is a semisimple infinite-dimensional Frobenius manifold of charge $d = 1$, with unit and Euler vector fields

$$e = (-1, 1), \quad E = \left(\lambda(z) - z\,\lambda'(z), \bar{\lambda}(z) - z\,\bar{\lambda}'(z)\right).$$

- The potential is

$$F = \frac{1}{2}\frac{1}{(2\pi i)^2}\oint\oint_{|z_1|<|z_2|}\frac{w(z_1)}{z_1}\frac{w(z_2)}{z_2}\log\frac{z_2 - z_1}{z_2}\,dz_1\,dz_2$$
$$+\frac{1}{2}\left(2v - \frac{1}{2\pi i}\oint_{|z|=1} w(z)\,\frac{dz}{z}\right)\left[\frac{1}{2\pi i}\oint_{|z|=1}\frac{w(z)}{z}\left(\log\frac{w(z)}{z} - 1\right)\,dz\right]$$
$$+\frac{1}{2}v^2 u + e^u\frac{1}{2\pi i}\oint_{|z|=1} w(z)\,\frac{dz}{z^2}, \qquad w(z) = \lambda(z) + \bar{\lambda}(z).$$

- F as a function of flat coordinates t_α, v, u solves WDVV equations.

Intersection form

- The intersection form on $M \setminus \Sigma$ is given by the formula

$$(\partial_1, \partial_2) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{\left(\frac{\partial_1 \lambda}{\lambda'} - \frac{\partial_1 \bar{\lambda}}{\bar{\lambda}'}\right)\left(\frac{\partial_2 \lambda}{\lambda'} - \frac{\partial_2 \bar{\lambda}}{\bar{\lambda}'}\right)}{\frac{\lambda}{\lambda'} - \frac{\bar{\lambda}}{\bar{\lambda}'}} \frac{dz}{z^2},$$

  where Σ is the locus in M defined by the conditions

$$\lambda'(z) \neq 0, \quad \bar{\lambda}'(z) \neq 0, \quad \lambda(z)\bar{\lambda}'(z) - \bar{\lambda}(z)\lambda'(z) \neq 0 \quad for \quad |z| = 1.$$

- The flat metric and the intersection form on M coincide with the flat metrics associated to the dispersionless limit of the Poisson pencil of 2D Toda (GC 2004).

Deformed flat connection of M

We find the explicit form of the deformed flat connection on 1-forms

$$\tilde{\nabla}_X \alpha = \partial_X \alpha - \Gamma_X \alpha - \zeta C_X(\alpha),$$
$$\tilde{\nabla}_{\frac{d}{d\zeta}} \alpha = \partial_\zeta \alpha - \mathcal{U}(\alpha) + \frac{1}{\zeta}\mathcal{V}(\alpha),$$

in terms of explicit operators on $T^*M = \mathcal{H}(S^1) \oplus \mathbb{C}^2$:

$$\mathcal{V}(\alpha) = \left(-\frac{\alpha(z)}{2} - z\alpha'(z)\left(\frac{w(z)}{zw'(z)}\right), -\frac{\alpha_v}{2}, \frac{\alpha_u}{2} \right),$$

$$\Gamma_X(\alpha) = \left(\frac{\alpha'(z)X(z)}{w'(z)}, 0, 0 \right), \quad \mathcal{U} = C_E,$$

$$\begin{aligned} C_X(\alpha) = \Big(& \frac{X(z)}{zw'(z)} \left((zw'(z)\alpha(z))_{>0} - (zw'(z))_{>0}\alpha(z) + z\alpha(z) + \frac{e^u}{z}(\alpha(z)+\alpha_v) + \alpha_u \right) + \\ & + (X_{>0}(z)\alpha(z))_{<0} + \left(X_{\leqslant 0}(z)\alpha(z) \right)_{\geqslant 0} + \frac{e^u}{z} X_u(\alpha(z)+\alpha_v) + X_v \alpha(z), \\ & (X(z)\alpha(z))_0 + X_u\alpha_u + X_v\alpha_v, \\ & \left(e^u (X(z) + zw'(z)X_u)(\alpha(z)+\alpha_v) \right)_0 - e^u X_u \alpha_v + X_v \alpha_u \Big). \end{aligned}$$

Solving the deformed flatness equations

A function y on M is deformed flat if, for $X \in TM$

$$\partial_X dy = (\Gamma_X + \zeta C_X) dy, \tag{a}$$

$$\partial_\zeta dy = (\mathcal{U} - \frac{1}{\zeta}\mathcal{V}) dy. \tag{b}$$

Lemma

For a function on M of the form

$$y(\lambda, \bar{\lambda}; \zeta) = \frac{\zeta^{-\frac{1}{2}}}{2\pi i} \oint_{|z|=1} F(\zeta\lambda(z), \zeta\bar{\lambda}(z)) \frac{dz}{z} + \phi(\zeta)$$

we have that (a) holds $\iff$ $F(x, \bar{x})$ satisfies

$$(F_{x\bar{x}} - F_{xx} - F_x)_{\geqslant -1} = 0, \quad (F_{\bar{x}\bar{x}} - F_{x\bar{x}} - F_{\bar{x}})_{\leqslant 1} = 0,$$
$$(F_{\bar{x}} - F_x - F)_0 = const, \quad \partial_u[e^u(F_{\bar{x}} - F_x - F)]_1 = 0.$$

Moreover: (a) $\implies$ (b).

Deformed flat coordinates

Solving the equations for $F(x, \bar{x})$ we obtain a set of deformed flat coordinates on M:

$$y_{\hat{\alpha}}(\lambda, \bar{\lambda}; \zeta) = \frac{\zeta^{-\frac{1}{2}}}{2\pi i} \oint_{|z|=1} F_{\hat{\alpha}}(\zeta\lambda(z), \zeta\bar{\lambda}(z)) \frac{dz}{z} + \phi_{\hat{\alpha}}(\zeta)$$

where

$$\begin{aligned}
F_{\alpha}(x, \bar{x}) &= -\frac{(x+\bar{x})^{(\alpha+1)}}{\alpha+1} \exp\left(\frac{\bar{x}-x}{2}\right) \qquad \text{for } \alpha \neq -1, \\
F_{-1}(x, \bar{x}) &= -\exp(-x)\left(\log\left(\frac{x+\bar{x}}{x}\right) + \mathrm{Ein}(-x) - 1\right) - \exp\left(\frac{\bar{x}-x}{2}\right), \\
F_{v}(x, \bar{x}) &= -\exp(-x)\left(\log\left(\frac{x+\bar{x}}{x}\right) + \mathrm{Ein}(-x) - 1\right) + \\
&\quad + \exp(\bar{x})\left(\log((x+\bar{x})\bar{x}) - \mathrm{Ein}(\bar{x}) - 1\right), \\
F_{u}(x, \bar{x}) &= \exp(\bar{x}) - 1.
\end{aligned}$$

Monodromy

- We have solved the operator-valued linear singular isomonodromic system on $\mathbb{C}$
$$\partial_\zeta Y = (\mathcal{U} + \frac{1}{\zeta}\mathcal{V})Y$$
where the fundamental "matrix" is in the normal Levelt form
$$Y = \Theta \zeta^{\mathcal{V}} \zeta^{R}$$
with $\Theta(\zeta)$ analytic in a neighborhood of $\zeta = 0$ and $\Theta(0) = 1$.
- $\mathcal{V}(\nabla t^{\hat{\alpha}}) = \mu_{\hat{\alpha}} \nabla t^{\hat{\alpha}}$ with $\mu_{\hat{\alpha}} \in \mathbb{Z}/2$ (highly resonant $\mathcal{V}$).
- R is the symmetric nilpotent operator on $TM = \mathcal{H}(S^1) \oplus \mathbb{C}^2$ given by
$$R(\nabla u) = 2\nabla v, \quad R(\nabla t^{\hat{\alpha}}) = 0 \quad \hat{\alpha} \neq u.$$

Hamiltonian densities of the principal hierarchy

The analytic part of the $y_{\hat{\alpha}}$ gives the following Levelt basis of deformed flat coordinates $\theta_{\hat{\alpha}}(\zeta)$:

$$\theta_\alpha(\zeta) = -\frac{1}{2\pi i}\oint_{|z|=1} \frac{(\lambda+\bar{\lambda})^{\alpha+1}}{\alpha+1} e^{\frac{\bar{\lambda}-\lambda}{2}\zeta}\ \frac{dz}{z} \quad \text{for} \quad \alpha \neq -1,$$

$$\theta_{-1}(\zeta) = -\frac{1}{2\pi i}\oint_{|z|=1} \left[e^{-\lambda\zeta}\left(\log\left(1+\frac{\bar{\lambda}}{\lambda}\right) + \mathrm{Ein}(-\lambda\zeta) - 1\right) + e^{\frac{\bar{\lambda}-\lambda}{2}\zeta}\right]\frac{dz}{z},$$

$$\theta_v(\zeta) = \frac{1}{2\pi i}\oint_{|z|=1} \Big[- e^{-\lambda\zeta}\left(\log\left(1+\frac{\bar{\lambda}}{\lambda}\right) + \mathrm{Ein}(-\lambda\zeta) - 1\right) + \\ + e^{\bar{\lambda}\zeta}\left(\log\bar{\lambda}(\lambda+\bar{\lambda}) - \mathrm{Ein}(\bar{\lambda}\zeta) - 1\right)\Big]\frac{dz}{z},$$

$$\theta_u(\zeta) = \frac{1}{2\pi i}\oint_{|z|=1} \frac{e^{\bar{\lambda}\zeta}-1}{\zeta}\ \frac{dz}{z}$$

Extended 2D Toda hierarchy

- The Hamiltonians of the principal hierarchy on $\mathcal{L}M$ are defined by

$$H_{\hat{\alpha},p} = \int_{S^1} \theta_{\hat{\alpha},p+1}\ dx \quad \text{where} \quad \theta_{\hat{\alpha}}(\zeta) = \sum_{p \geqslant 0} \theta_{\hat{\alpha},p} \zeta^p$$

  and the Hamiltonian flows by

$$\frac{\partial}{\partial t_{\hat{\alpha},p}} \cdot = \{\cdot, H_{\hat{\alpha},p}\}_1.$$

- bi-Hamiltonian recursion relations :

$$\{\cdot, H_{\alpha,p-1}\}_2 = (p+\alpha+1)\{\cdot, H_{\alpha,p}\}_1,$$
$$\{\cdot, H_{u,p-1}\}_2 = (p+1)\{\cdot, H_{u,p}\}_1,$$
$$\{\cdot, H_{v,p-1}\}_2 = p\{\cdot, H_{v,p}\}_1 + 2\{\cdot, H_{u,p-1}\}_1.$$

- The Hamiltonians of the dispersionless 2D Toda hierarchy are finite combinations of the Hamiltonians of the principal hierarchy.

Lax formulation

The principal 2D Toda hierarchy admits a Lax formulation

$$\frac{\partial \lambda}{\partial t^{\hat{\alpha},p}} = \{-(A_{\hat{\alpha},p})_-, \lambda\}, \quad \frac{\partial \bar{\lambda}}{\partial t^{\hat{\alpha},p}} = \{(A_{\hat{\alpha},p})_+, \bar{\lambda}\}$$

for $\hat{\alpha} \in \mathbb{Z} \cup \{u, v\}$, where

$$\begin{aligned}
A_{\alpha,p} &= -\frac{(\lambda+\bar{\lambda})^{\alpha+1}}{\alpha+1}\frac{1}{p!}\Big(\frac{\bar{\lambda}-\lambda}{2}\Big)^p, \qquad \alpha \neq -1 \\
A_{-1,p} &= -\frac{(-\lambda)^p}{p!}\Big(\log\frac{\lambda+\bar{\lambda}}{\lambda} + c_p - 1\Big) - \frac{1}{p!}\Big(\frac{\bar{\lambda}-\lambda}{2}\Big)^p, \\
A_{v,p} &= -\frac{(-\lambda)^p}{p!}\Big(\log\frac{\lambda+\bar{\lambda}}{\lambda} + c_p - 1\Big) + \\
&\quad + \frac{\bar{\lambda}^p}{p!}\Big(\log(\lambda+\bar{\lambda})\bar{\lambda} - c_p - 1\Big), \quad A_{u,p} = \frac{\bar{\lambda}^{p+1}}{p+1!}
\end{aligned}$$

where $c_l = 1 + \cdots + \frac{1}{l}$ and $c_0 = c_{-1} = 0$ are the harmonic numbers.
(for $A_{\alpha,p}$ cf. Adler, van Moerbeke 1997)