Note Title 13/12/2011

Geometrical Methods in Mathematical Physics

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Recursion operators and Frobenius manifolds

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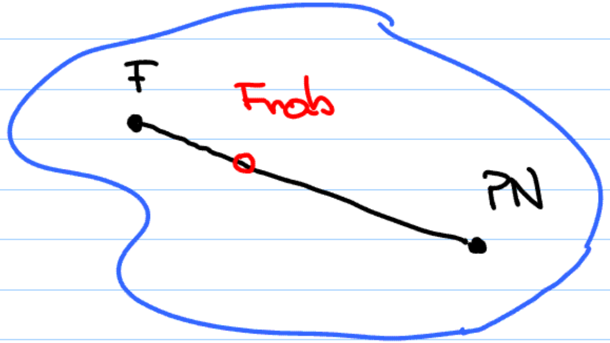
In this talk I will present a comparative study of three different types of manifolds:

- bihamiltonian manifolds
- Frobenius manifolds
- F-manifolds

with the aim to build a bridge
between them and to create a
perspective allowing to study them
in a unified way

To reach this goal my strategy will be to show that, in the category of smooth manifolds, there is a single and simple path which joins the category of F-manifolds to the category of bihamiltonian manifolds passing through the category of Frobenius manifolds:



$$\boxed{F \supset H_1 = H \supset H_2 \supset \cdots \supset H_n \supset \cdots \supset H_\infty \supset PN}$$

F : F-manifolds

PN : Poisson-Nijenhuis manifolds

$$PN \subset \text{Frobenius} \subset F.$$

Step 1: H_1 and H_∞

structure: $\kappa : TM \to TM$

$X : M \to TM$

$\theta : M \to T^*M$

axioms:

H_∞	$\mathrm{Torsion}(\kappa) = 0$	$\mathrm{Lie}_X(\kappa) = 0$	$d\theta = 0$ $d(\kappa\theta) = 0$
H_1	$\mathrm{Haantjes}(\kappa) = 0$	$\mathrm{Lie}_X(\kappa) = 0$	$d\theta = 0$ $d(\kappa\theta) = 0$ $\theta(T_\kappa(X,Y)) = 0$

$$\begin{array}{c} \theta \\ K \downarrow \\ \theta' \\ K \downarrow \\ \theta'' \end{array}$$

short Lenard chain : $d\theta = 0 \quad d\theta' = 0 \quad d\theta'' = 0$

$K_0 = Id \quad K_1 = K$

H_∞	$\mathrm{Torsion}(K_j) = 0$	$\mathrm{Lie}_X(K_j) = 0$	$d(K_j K_e \theta) = 0 \quad j = 0, 1$
H_1	$\mathrm{Haantjes}(K_j) = 0$	$\mathrm{Lie}_X(K_j) = 0$	$d(K_j K_e \theta) = 0 \quad j = 0, 1$

F and PN :

$$\boxed{F \supset H_1 \supset \cdots\cdots\cdots \supset H_\infty \supset PN}$$

$F := H_1$ minus θ : F-manifolds

$PN := H_\infty$ plus Poisson : Poisson-Nijenhuis man.

Step 2 : H_2

structure:

$K_1 : TM \to TM$

$K_2 : TM \to TM$

$X : M \to TM$

$\theta : M \to T^*M$

axioms:

$[K_j, K_e] = 0$	$\mathrm{Haantjes}(K_j) = 0$	$\mathrm{Lie}_X(K_j) = 0$	$d(K_j K_e \theta) = 0$	$j = 0, 1, 2$

$$\begin{tikzcd}
 & dA \arrow[dl, "K_1"'] \arrow[dr, "K_2"] & & \\
dB \arrow[d, "K_1"'] \arrow[dr] & & dC \arrow[dl] \arrow[dr, "K_2"] & \\
dP & dQ & & dR
\end{tikzcd}$$

new concept: compatibility of two short Lenard chains

Step 3 : H_n

structure :

$K_1 : TM \to TM$
$K_2 : TM \to TM$
$\vdots$
$K_n : TM \to TM$

$X : M \to TM$
$\theta : M \to T^*M$

axioms :

$[K_j, K_l] = 0$	$\text{Haantjes}(K_j) = 0$	$\text{Lie}_X(K_j) = 0$	$d(K_j K_l \theta) = 0$ $\quad j = 0, 1, 2 \ldots n$

$H_n \supset H_\infty$:

if Torsion$(K) = 0$ then $K_1 = K$ $K_2 = K^2 \dots K_n = K^n$

The long Lenard chain of the theory of bihamilto. nian manifolds is the composition of infinitely many short Lenard chains defined by the powers of K

FROBENIUS MANIFOLDS

A manifold M of $\dim M = n$ is a Frobenius manifold iff it belongs to $\mathcal{H}_{n-1}$:

commutability : $[k_j, k_\ell] = 0$

Haantjes : $\text{Haantjes}(k_j) = 0$

symmetry : $\text{Lie}_X (k_j) = 0$

compatibility : $d(k_j k_\ell \vartheta) = 0$ $\qquad j = 0, 1, \dots, n-1$

and

$$k_0 X \wedge k_1 X \wedge \cdots \wedge k_{n-1} X \neq 0$$

$$k_0 \vartheta \wedge k_1 \vartheta \wedge \cdots \wedge k_{n-1} \vartheta \neq 0$$

$$\underset{X}{\mathrm{Lie}}(\vartheta) = 0$$

$$F \supset H_1 \supset H_2 \cdots \supset \underset{\substack{\uparrow \\ \text{Frobenius}}}{H_{n-1}} \supset \cdots \supset H_\infty = N \supset PN$$

coordinates:

$$\frac{\partial}{\partial x^i} = K_i X$$

$$dA_i = K_i \theta$$

hierarchy of moments:

$$\vartheta_i = \langle \vartheta | k_j | X \rangle$$ Eq. of 1-form

$$g_{je} = \langle \vartheta | k_j k_e | X \rangle$$ metric

$$c_{jem} = \langle \vartheta | k_j k_e k_m | X \rangle$$ 3-points correlation function

. . . .

$\square$

commutative algebra

$$K_j K_e = C^m_{je} K_m$$

Kodaira-Spencer mapping

$$\begin{tikzcd}
\langle \kappa_0, \kappa_1, \ldots, \kappa_{n-1} \rangle \arrow[d, bend left=60, "\boxed{ev : \frac{\partial}{\partial x_j} = \kappa_j X}"] \\
TM \arrow[u, bend left=60, "\boxed{KS = ev^{-1}}"]
\end{tikzcd}$$

multiplication on TM :

$$\begin{tikzcd}
K_j \arrow[r] & K_j \cdot K_\ell & K_\ell \arrow[l] \\
\partial_j \arrow[u, "KS"] \arrow[r] & \partial_j \circ \partial_k \arrow[u, leftarrow, "ev"'] & \partial_\ell \arrow[l] \arrow[u, "KS"']
\end{tikzcd}$$

Construction of a short Lenard chain: an example

$$
\begin{tikzcd}[column sep=small]
 & & dA \arrow[dl, "K_1"'] \arrow[dr, "K_2"] & & \\
 & dB \arrow[dl] \arrow[dr] & & dC \arrow[dl] \arrow[dr] & \\
dP & & dQ & & dR
\end{tikzcd}
$$

short Lenard chain

in $\dim M = 3$

$$
\begin{aligned}
K_1\, dA &= dB & \qquad K_2\, dA &= dC \\
K_1\, dB &= dP & \qquad K_2\, dB &= dQ \\
K_1\, dC &= dQ & \qquad K_2\, dC &= dR
\end{aligned}
$$

(A, B, C) = coordinates

(P, Q, R) = functions of A, B, C

Commutativity : $k_1 \circ k_2 - k_2 \circ k_1 = 0$

$$\frac{\partial P}{\partial A} = \frac{\partial P}{\partial C}\left(\frac{\partial Q}{\partial B} - \frac{\partial R}{\partial C}\right) + \frac{\partial Q}{\partial C}\left(\frac{\partial Q}{\partial C} - \frac{\partial P}{\partial B}\right)$$

$$\frac{\partial Q}{\partial A} = \frac{\partial P}{\partial C}\frac{\partial R}{\partial B} - \frac{\partial Q}{\partial C}\cdot\frac{\partial Q}{\partial B}$$

$$\frac{\partial R}{\partial A} = \frac{\partial Q}{\partial B}\left(\frac{\partial Q}{\partial B} - \frac{\partial R}{\partial C}\right) + \frac{\partial R}{\partial B}\left(\frac{\partial Q}{\partial C} - \frac{\partial P}{\partial B}\right)$$

symmetry X : $\qquad \mathcal{L}_X(K_j) = 0 \qquad X \wedge K_1 X \wedge K_2 X \neq 0$

$$\begin{aligned} P &= C + \Phi(A,B) \\ Q &= \quad\ \Psi(A,B) \\ R &= \quad\ \chi(A,B) \end{aligned} \quad \Rightarrow \quad X = \frac{\partial}{\partial C}$$

equazione WDVV = commutatività + separazione

$$\boxed{\begin{aligned}
\frac{\partial P}{\partial A} &= \frac{\partial P}{\partial C}\left(\frac{\partial Q}{\partial B}-\frac{\partial R}{\partial C}\right)+\frac{\partial Q}{\partial C}\left(\frac{\partial Q}{\partial C}-\frac{\partial P}{\partial B}\right)\\
\frac{\partial Q}{\partial A} &= \frac{\partial P}{\partial C}\frac{\partial R}{\partial B}-\frac{\partial Q}{\partial C}\frac{\partial Q}{\partial B}\\
\frac{\partial R}{\partial A} &= \frac{\partial Q}{\partial B}\left(\frac{\partial Q}{\partial B}-\frac{\partial R}{\partial C}\right)+\frac{\partial R}{\partial B}\left(\frac{\partial Q}{\partial C}-\frac{\partial P}{\partial B}\right)
\end{aligned}} \longrightarrow \boxed{\begin{aligned}
\phi_A &= \psi_B\\
\psi_A &= \chi_B\\
\chi_A &= \psi_B^2-\chi_B\phi_B
\end{aligned}}$$

$$\downarrow$$

$$\boxed{F_{AAA} = F_{ABB}^2 - F_{AAB}\cdot F_{BBB} \quad\Big|\quad \phi = F_{BB} \quad \psi = F_{AB} \quad \chi = F_{AA}}$$

formal solution :

$$F = \frac{1}{2} B^2 C + \sum_{k=1}^{\infty} \Phi_k(B) \frac{A^{3k-1}}{(3k-1)!}$$

C is separate

A is the generator of the Lenard chain.

$$\Phi_2(B) = -\Phi_1'''(B)\,\Phi_1'(B) + 2\,\Phi_1''(B)\,\Phi_1''(B)$$

$$\Phi_3(B) = -\Phi_2'''(B)\,\Phi_1'(B) + 10\,\Phi_1''(B)\,\Phi_2''(B) - 10\,\Phi_1'''(B)\,\Phi_2'(B)$$

$$\Phi_4(B) = -\Phi_3'''(B)\,\Phi_1'(B) + 16\,\Phi_1''(B)\,\Phi_3''(B) + 70\,\Phi_2''(B)^2 - 28\,\Phi_1'''(B)\cdot\Phi_3'(B)$$

(remarkable property: integer coefficients) if $\Phi_1(B) = B^2$; $\Phi_2 = 8$

$\Phi_3 = 0$ Saito: A_3

$\Phi_4 = 0$

if $\Phi_1(B) = e^B$; $\Phi_2(B) = 1 \cdot e^B$

$\Phi_3(B) = 12 \cdot e^B$ Kontsevich-Manin

$\Phi_4(B) = 620 \cdot e^B$

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This example shows that one can arrive to the GW invariants also through the study of short Lenard chains defined by the basic equation

$$d(K_j K_e \theta) = 0$$

$\blacksquare$