Conservation laws, hodograph transformation and boundary value problems of plane plasticity

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Outline

Introduction

* The Riemann method is widely used for solution of Cauchy problem for hyperbolic linear equation for one function of two independent variables. The real reason for the introduction of adjoint equation to obtain Riemann function is to make available the line integral vanishes around closed paths [6]. In other words there is a conservation law of the special form.
* The same method is applied for linear system obtained from quasi-linear one by hodograph transformation. This linearization is valid only when the Jacobian of transformation is not equal to zero, that is unknown before the solution of the system.
* Conservation laws of quasi-linear homogeneous hyperbolic system for two functions of two independent variables, related with the solution of corresponding linearized system are used in the present talk for the solution of the Cauchy problem. The vanishing of Jacobian is not a restriction now, that allows to construct the characteristic fields including the simple waves.
* Some applications for plane plasticity and gas dynamics are considered. In particular the Mikhlin problem for the loaded cavity is solved for any convex form of contour.

Hyperbolic quasi-linear system

Let us consider a quasilinear system of homogeneous PDEs of two independent variables x, y and two dependent ones u, v in the form [10]:

$$A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0, \qquad (1)$$

where $A = \|a_{ij}(u,v)\|$, $B = \|b_{ij}(u,v)\|$, $i,j = 1,2$, $U = (u,v)^T$.
If matrix A is not degenerate, then system (1) can be written in the normal form

$$\frac{\partial U}{\partial x} + M\frac{\partial U}{\partial y} = 0, \qquad (2)$$

were $M = \|m_{ij}(u,v)\|$.

Let system (2) be a strictly hyperbolic one. It means that matrix M has two real different eigenvalues λ_1 and λ_2, obtained as a roots of the equation:

$$\det(M - \lambda E) = 0: \quad 2\lambda_{1,2} = m_{11} + m_{22} \pm \sqrt{(m_{11} - m_{22})^2 + 4m_{12}m_{21}},$$

that gives two eigenvectors $l_1 = \left(l_1^1, l_1^2\right)$ and $l_2 = \left(l_2^1, l_2^2\right)$ respectively.

Let us consider the differential forms

$$\omega_k = l_k^1(u,v)du + l_k^2(u,v)dv = 0, \quad k = 1,2,$$

which can be integrated, because in this case there exist an integrating factors always. Then, the corresponding two integrals $\Phi_k(u,v) = \text{const}$ can be taking as a Riemann invariants $r_k = \Phi_k(u,v)$ and the system (2) takes the diagonal form:

$$\frac{\partial R}{\partial x} + \Lambda \frac{\partial R}{\partial y} = 0, \ \Lambda = \text{diag}(\lambda_1, \lambda_2), \ R = (r_1, r_2)^T. \qquad (3)$$

System (3) has two families of real characteristic curves determined by the following equations:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda_1(u,v), \ \frac{\mathrm{d}y}{\mathrm{d}x} = \lambda_2(u,v), \ \lambda_1 \neq \lambda_2, \qquad (4)$$

that is the variable r_1 is the invariant along the first characteristic curve and r_2 is the invariant along the second one.

It is well known, that system (2) can be linearized by a so-called *hodograph transformation* of the form $x = x(u, v)$, $y = y(u, v)$. This 'speedgraph' transformation [7] is just an interchange of roles of the unknown functions and the independent variables. Thus, the system (1) takes the linear form:

$$\det(A)(A^T)^{-1}\nabla_U y(u, v) = \det(B)(B^T)^{-1}\nabla_U x(u, v),\ \nabla_U = (\partial/\partial u, \partial/\partial v)^T ,$$

and for the diagonal form (3) one can obtain

$$\Lambda\nabla_R y(r_1, r_2) = \det(\Lambda)\nabla_R x(r_1, r_2),\ \nabla_R = (\partial/\partial r_1, \partial/\partial r_2)^T . \tag{5}$$

The system of such a form are widely used in the mechanics of a continuum media [10], for example in the gas dynamics for describing isoentropic plane-symmetry flows; in the theory of plane plasticity for the stresses of a deformed region under the different yield criterion [4], for the motion of granular materials, for the propagation of the plane wave of loading in homogeneous semi-infinite elastic-plastic beam [8], etc.

Let us note, that it is possible to obtain the solution of system (3) from the solution of (5) and vice versa only when two corresponding Jacobians $J_1 = \partial(r_1, r_2)/\partial(x, y)$ and $J_2 = \partial(x, y)/\partial(r_1, r_2)$ are not equal to zero:

$$J_1 = \frac{\partial r_1}{\partial y}\frac{\partial r_2}{\partial x} - \frac{\partial r_2}{\partial y}\frac{\partial r_1}{\partial x} = J_2^{-1}. \tag{6}$$

System (2) can be extended for three or more functions and in this case is called one-dimensional system of hydrodynamic type [15]. A natural hamiltonian formalism was proposed for this class of homogeneous systems of PDE and the generalized hodograph method generates from every its solution a symmetry (commuting flow) that finally leads to solution of (2).

Let us set up the **Cauchy problem** for the system (2) due to [10]: in some neighborhood of the arc C: $a \leqslant \tau \leqslant b$ of an initial curve $L = \{(x, y) : x = x(\tau), y = y(\tau)\}$ it is necessary to determine the solution of (2) satisfying on L the initial condition

$$U(x(\tau), y(\tau)) = U^0(\tau), \ \tau \in [a, b]. \tag{7}$$

Conservation Laws

Let us seek the conservation law of system (3) directly in the form

$$\frac{\partial}{\partial x}\varphi(u,v)+\frac{\partial}{\partial y}\psi(u,v)=0, \tag{8}$$

which is valid for any solution of (3). Let us multiply (3) by a vector $\alpha=(\alpha_1,\alpha_2)$ [10], then, eliminating α, we obtain the linear system for the functions φ and ψ:

$$\Lambda\nabla_R\varphi(r_1,r_2)=\nabla_R\psi(r_1,r_2). \tag{9}$$

Now let us describe how to solve Cauchy problem using conservation laws. Let $P(x(a),y(a))$, $Q(x(b),y(b))$ be two end-points of the arc C, $M(M_x,M_y)$ be a point of intersection of two characteristic lines: $r_1=r_1^0$, going from the point Q and $r_2=r_2^0$, going from the point P.

Taking integral over the closed path PQM, which due to Green theorem is equal to zero, we have:

$$\oint_{PQM} \psi dx - \varphi dy = 0 = \int_{PQ} + \int_{r_1=r_1^0} (\psi - \varphi\lambda_1)\, dx + \int_{r_2=r_2^0} (\psi - \varphi\lambda_2)\, dx\,. \qquad (10)$$

Integrating by parts the above two integrals and taking (without loss of generality) the following conditions:

$$(\psi - \lambda_1\varphi)|_{r_1=r_1^0} = 1,\ \ (\psi - \lambda_2\varphi)|_{r_2=r_2^0} = 0, \qquad (11)$$

we get the coordinate M_x

$$M_x = x(b) - \int_{PQ} \left(\psi \frac{dx}{d\tau} - \varphi \frac{dy}{d\tau}\right) d\tau. \qquad (12)$$

Analogically, for the y-coordinate of the point M we obtain:

$$M_y = y(b) - \int\limits_{PQ} \left(\psi \frac{dx}{d\tau} - \varphi \frac{dy}{d\tau} \right) d\tau, \tag{13}$$

but now the conditions for the functions φ, ψ are the following ones:

$$(\psi/\lambda_1 - \varphi)|_{r_1 = r_1^0} = 1, \ \ (\psi/\lambda_2 - \varphi)|_{r_2 = r_2^0} = 0. \tag{14}$$

In such a way, the coordinates of the point M are known and one can reconstruct the values of functions u, v from the initial condition (7) .

In other words the Cauchy problem for quasi-linear system (2) with (7) is reduced to two Cauchy problems for the linear one (9) with conditions (11) and (14). To determine the functions φ and ψ for these problems one can use the same Riemann method.

Note, that the above procedure can be used to solve Goursat problem, when the initial data are prescribed on two intersecting characteristics (see [12]).

Comparing equations (9) with system (5) one can obtain the linearization of (2) without any mention of the vanishing of Jacobian J_1 and calculate the conservation laws directly in two cases:

1. if $\det(\Lambda) = K = \text{const}$, then $\varphi = y(r_1, r_2)$, $\psi = Kx(r_1, r_2)$;
2. if $\lambda_1 = -\lambda_2$, then $\varphi = x(r_1, r_2)$, $\psi = -y(r_1, r_2)$.

An example of the first case is the system of the ideal plane plasticity [4]:

$$M = \begin{pmatrix} -\cot 2v & -2k/\sin 2v \\ -1/(2k\sin 2v) & -\cot 2v \end{pmatrix}, \qquad (15)$$

with $\det(\Lambda) = -1$.

One of the example of the system (2) for the second case mentioned above, is the system

$$m_{11} = m_{22} = 0,\ m_{12} = -1,\ m_{21} = F^2(u), \qquad (16)$$

related with the well known equation

$$\frac{\partial^2 Y}{\partial x^2} = F^2\left(\frac{\partial Y}{\partial y}\right)\frac{\partial^2 Y}{\partial y^2},$$

investigated in Ref. [17] in the context with Fermi, Pasta and Ulam (1955) results on the vibration of a nonlinear, loaded (or beaded) finite string. In the case (16) we have $\lambda_1 = -\lambda_2 = -F$.

Another possibility to find simple relation between conservation laws and solution of linearized system is to use well-known lemma about Laplace invariants (h, k) for two equivalent hyperbolic equations [9], namely:

"two hyperbolic equations of the form

$$\frac{\partial^2 V(r_1, r_2)}{\partial r_1 \partial r_2} + C_1(r_1, r_2)\frac{\partial V(r_1, r_2)}{\partial r_1} + C_2(r_1, r_2)\frac{\partial V(r_1, r_2)}{\partial r_2} + C_3(r_1, r_2)V(r_1, r_2) = 0$$

are equivalent up two the factor-function of r_1, r_2 iff its corresponding Laplace invariants:

$$h = \frac{\partial C_1}{\partial r_1} + C_1C_2 - C_3,\ k = \frac{\partial C_2}{\partial r_2} + C_1C_2 - C_3 \tag{17}$$

are equal".

Expressing (5) and (9) in terms of the hyperbolic equations of the second order:

$$\frac{\partial^2 x}{\partial r_1 \partial r_2} - \frac{1}{\lambda_1 - \lambda_2}\frac{\partial \lambda_2}{\partial r_2}\frac{\partial x}{\partial r_1} + \frac{1}{\lambda_1 - \lambda_2}\frac{\partial \lambda_1}{\partial r_1}\frac{\partial x}{\partial r_2} = 0, \tag{18}$$

$$\frac{\partial^2 \varphi}{\partial r_1 \partial r_2} + \frac{1}{\lambda_1 - \lambda_2}\frac{\partial \lambda_1}{\partial r_2}\frac{\partial \varphi}{\partial r_1} - \frac{1}{\lambda_1 - \lambda_2}\frac{\partial \lambda_2}{\partial r_1}\frac{\partial \varphi}{\partial r_2} = 0 \tag{19}$$

and comparing its Laplace invariants:

$$h_x = -\frac{1}{\lambda_1 - \lambda_2}\frac{\partial^2 \lambda_2}{\partial r_1 \partial r_2} - \frac{1}{(\lambda_1 - \lambda_2)^2}\frac{\partial \lambda_2}{\partial r_1}\frac{\partial \lambda_2}{\partial r_2} = k_\varphi,$$

$$k_x = \frac{1}{\lambda_1 - \lambda_2}\frac{\partial^2 \lambda_1}{\partial r_1 \partial r_2} - \frac{1}{(\lambda_1 - \lambda_2)^2}\frac{\partial \lambda_1}{\partial r_1}\frac{\partial \lambda_1}{\partial r_2} = h_\varphi$$

one can obtain the relation between eigenfunctions:

$$(\lambda_1 - \lambda_2)\frac{\partial^2}{\partial r_1 \partial r_2}(\lambda_1 + \lambda_2) = \frac{\partial \lambda_1}{\partial r_1}\frac{\partial \lambda_1}{\partial r_2} - \frac{\partial \lambda_2}{\partial r_1}\frac{\partial \lambda_2}{\partial r_2} \tag{20}$$

when there exist the function $w(r_1, r_2)$, giving the simple relation between solution of (18) and (19): $\varphi = wx$.

In such a way we obtain the solution of Cauchy problem for the original system (2), (7) without any restrictions on the Jacobian of hodograph transformation.

1. Plane ideal plasticity with Saint-Venant–Mises yield criterion
This system was investigated, using the group of admitted symmetries: for its invariant solutions see [1] and for the reproduction of solutions see [13, 14, 16]. Moreover, all its conservation laws and highest symmetries were described in [11]. Being semi-inverse method, group analysis provides analytical solutions and then one can determine the boundary conditions for obtained solutions. But if the the boundary conditions are given from the beginning, then the method of conservation laws can be applied to solve boundary problem directly.
The system has the form:

$$\begin{aligned} &\frac{\partial \sigma}{\partial x} - 2k\left(\frac{\partial \theta}{\partial x}\cos 2\theta + \frac{\partial \theta}{\partial y}\sin 2\theta\right) = 0, \\ &\frac{\partial \sigma}{\partial y} - 2k\left(\frac{\partial \theta}{\partial x}\sin 2\theta - \frac{\partial \theta}{\partial y}\cos 2\theta\right) = 0, \end{aligned} \tag{21}$$

where σ is a hydrostatic pressure, θ is an angle between the first main direction of a stress tensor and the ox-axis, k is a constant of plasticity.

The functions for this system in the form (3) are the following:

$$u = \sigma,\ v = \theta; \lambda_1 = \tan v,\ \lambda_2 = -\cot v;\ r_1 = \frac{u}{2k} - v,\ r_2 = \frac{u}{2k} + v. \qquad (22)$$

Solution of problem (9), (11) has a form [12]

$$\varphi = 2\frac{\partial\rho}{\partial r_1}\cos v - \rho\sin v,\ \psi = 2\frac{\partial\rho}{\partial r_1}\sin v + \rho\cos v, \qquad (23)$$

where function $\rho(r_1, r_2)$ looks like this

$$\rho(r_1, r_2) = R\left(r_1, r_1^0, r_2, r_2^0\right)\cos\left(\frac{r_2^0 - r_1^0}{2}\right) - \frac{1}{2}\int\limits_{r_2^0}^{r_2} R\left(r_1, r_1^0, r_2, \tau\right)\sin\left(\frac{\tau - r_1^0}{2}\right) d\tau.$$

Accordingly, the solution of the problem (9), (14) is

$$\rho(r_1, r_2) = R\left(r_1, r_1^0, r_2, r_2^0\right)\sin\left(\frac{r_2^0 - r_1^0}{2}\right) + \frac{1}{2}\int\limits_{r_2^0}^{r_2} R\left(r_1, r_1^0, r_2, \tau\right)\cos\left(\frac{\tau - r_1^0}{2}\right) d\tau,$$

$R\left(r_1, r_1^0, r_2, r_2^0\right) = I_0\left(\sqrt{(r_1 - r_1^0)(r_2 - r_2 0)}\right)$ is the Bessel function of a zero order of imaginary argument, $I_0(0) = 1, {I_0}'(0) = 0$.

In this case $\det(\Lambda) = -1$, so $\varphi = y(r_1, r_2)$, $\psi = -x(r_1, r_2)$.

2. Nonlinear hyperbolic heat equation

As indicated in [2], the hyperbolic heat equation

$$\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial t}+\frac{U}{\tau_0}\right)-\frac{\partial}{\partial X}\left(\frac{\chi_0^2}{U^2}\frac{\partial U}{\partial X}\right)=0, \tag{24}$$

where χ_0 and τ_0 are positive constants can be expressed in the form of the following quasilinear system

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0,\ \frac{\partial v}{\partial x}-\frac{\chi_0^2}{u^2}\frac{\partial u}{\partial y}=0 \tag{25}$$

by introducing the potential function $u(x,y)$ and setting

$$v=\tau_0 e^{t/\tau_0}U, y=X, x=e^{t/\tau_0}.$$

In this case

$$r_{1,2}=ue^{\mp v^2/\chi_0},\ \lambda_1=-\lambda_2=\frac{\chi_0}{\sqrt{r_1r_2}},$$

so $\varphi=x(r_1,r_2)$, $\psi=-y(r_1,r_2)$.

3. Gas dynamics

The one-dimensional isentropic flow of polytropic gas in Euler coordinates in case of plane symmetries as it well known [10] is described by the following hyperbolic system:

$$\frac{\partial s}{\partial t} + (\alpha s + \beta r)\frac{\partial s}{\partial x} = 0, \ \frac{\partial r}{\partial t} + (\alpha r + \beta s)\frac{\partial r}{\partial x} = 0, \qquad (26)$$

where $s = r_1$, $r = r_2$ are the Riemann invariants, so

$$\lambda_1 = \alpha r_1 + \beta r_2, \ \lambda_2 = \alpha r_2 + \beta r_1,$$

$\alpha = 1/2 + (\gamma - 1)/4$, $\beta = 1/2 - (\gamma - 1)/4$, $\gamma = \text{const} \neq 1$ is a parameter of polytrope.

In this case, equation (20) is satisfied, because $h_x = h_\varphi = -\alpha\beta$, so there is a relation between the conservation laws and solution of linearized system of the form $\varphi = w(r_1, r_2)x(r_1, r_2)$.

Explicitly, the solution of the problem (9), (11) has the form:

$$\varphi = \frac{\lambda_1\rho_1 - \lambda_2\rho_2}{\lambda_1 - \lambda_2},\ \psi = \frac{\rho_1 - \rho_2}{\lambda_1 - \lambda_2}, \tag{27}$$

where

$$\rho_1 = \rho_2 - \frac{r_2 - r_1}{K}\frac{\partial\rho_2}{\partial s},\ K = \frac{\gamma + 1}{2(1-\gamma)},$$

$$\rho_2(r_1, r_2) = (r_2^0 - r_1)^{-K}(r_2 - r_1^0)^{-(K+1)}(r_2^0 - r_1^0)^{2K+1}F(r_1^0, r_2^0; r_1, r_2) +$$
$$+(K+1)(r_2 - r_1^0)^{-(K+1)}\int_{r_2^0}^{r_2}(t - r_1)^{-K}(t - r_1^0)^{2K}F(r_1^0, t; r_1, r_2)dt, \tag{28}$$

and $F(r_1^0, r_2^0; r_1, r_2) = {}_2F_1\left(K, K+1; 1; \frac{(r_1^0 - r_1)(r_2 - r_2^0)}{(r_2 - r_1^0)(r_2^0 - r_1)}\right)$ is a hypergeometric function.

Analogically, for the problem (9), (14) the function ρ_2 has a little modification:

$$\rho_2(r_1, r_2) = (\beta r_2^0 + \alpha r_1^0)(r_2^0 - r_1)^{-K}(r_2 - r_1^0)^{-(K+1)}(r_2^0 - r_1^0)^{2K+1} F(r_1^0, r_2^0; r_1, r_2) +$$

$$+(r_2 - r_1^0)^{-(K+1)} \int_{r_2^0}^{r_2} (t - r_1)^{-K}(t - r_1^0)^{2K+1} F(r_1^0, t; r_1, r_2) \left(\beta + \frac{K+1}{t - r_1^0}(\beta t + \right.$$

$$\left. + \alpha r_1^0)\right) dt.$$

4. Loaded homogeneous semi-infinite elastic-plastic beam

The process of propagation of plastic deformations in semi-infinite elastic-plastic beam, dynamically loaded on the one end in Lagrange coordinates is described by the following system [8]:

$$\rho\frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x}, \ \frac{\partial v}{\partial x} = \frac{1}{\rho a^2(\sigma)}\frac{\partial \sigma}{\partial t}, \tag{29}$$

where $\rho = \text{const}$ is a density, $v(t, x)$ is a velocity of medium particles, a tension $\sigma = \sigma(\varepsilon)$ is monotonically increasing function of the deformation $\varepsilon(x, t)$. Introducing the function $u(x, t)$, such that $v = \frac{\partial u}{\partial t}$, $\varepsilon = \frac{\partial u}{\partial x}$ we come to nonlinear wave equation:

$$\frac{\partial^2 u}{\partial t^2} - a^2(\varepsilon)\frac{\partial^2 u}{\partial x^2} = 0,$$

so $a(\varepsilon)$ is a speed of the longitudinal wave propagation in the beam.

Eigenfunctions have the form $\lambda_{1,2} = \mp a(\sigma)$ and Riemann invariants are the following ones:

$$r_{1,2} = v \pm \int\limits_0^\sigma \frac{d\sigma_1}{a(\sigma_1)}. \qquad (30)$$

This is the case when $\lambda_1 = -\lambda_2$, then $\varphi = t(r_1, r_2)$, $\psi = -x(r_1, r_2)$.
In the particular case if $a = \sqrt{\sigma}$, then

$$\lambda_{1,2} = \mp\sqrt{\sigma};\ r_{1,2} = v \pm 2\sqrt{\sigma}$$

and system (9) can be reduced to Euler-Poisson-Darboux equation in the form

$$\frac{\partial^2\varphi}{\partial r_1 \partial r_2} + \frac{2}{r_1 - r_2}\left(-\frac{\partial\varphi}{\partial r_1} + \frac{\partial\varphi}{\partial r_2}\right) = 0 \qquad (31)$$

with the well known Riemann function.

In the work [5] the Mikhlin problem for the cavity in an infinite medium loaded by a constant shear stress in addition to a uniform pressure is solved under the condition $J_1 \neq 0$. The using of conservation laws permits forget about this condition.
Let us consider one example, namely let the contour be given by the 2π-periodic curve:

$$\begin{cases} x = -r\cot t,\ y = -r, \quad t \in [\gamma - \pi, -\gamma), \\ x = -r\cot t,\ y = r, \quad t \in (\gamma, \pi - \gamma), \\ x = r\cos\dfrac{\pi t}{2\gamma},\ y = r\sin\dfrac{\pi t}{2\gamma}, \quad t \in [-\gamma, \gamma], \\ x = r\cos\dfrac{\pi t}{2(\pi-\gamma)} - a,\ y = r\sin\dfrac{\pi t}{2(\pi-\gamma)}, \ t \in (\gamma - \pi, \pi - \gamma], \end{cases}$$

where a is the distance form $(0,0)$ to the center of curve parts with radio r, $\gamma = \arctan(r/a)$. This contour is loaded by the normal and tangent stresses:

$$\sigma_n = -p, \tau_n = 0,$$

where n is the normal to the contour.

Let us define as $N(t) = \arctan \dfrac{y_t'}{x_t'}$ the angle between the tangent to the contour and x-axes. Putting $p = k = 1/2$, then the initial conditions take the form

$$\sigma|_L = -1, \ \ \theta|_L = \begin{cases} N(t) - \pi/4 + \pi/2, & t \in (0, \pi), \\ N(t) - \pi/4 + 3\pi/2, & t \in (\pi, 2\pi), \\ -\pi/4, & t = 0, \\ 3\pi/4, & t = \pi. \end{cases}$$

The solution of this problem is given on the Figure (for $a = 4$, $r = 3$), where the first family of the characteristic curves is constructed, using conservation laws described in subsection 1.

Let us note, that the stress field near the stright-line border of the cavity can not be constructed as a solution of linearized system because in this domain we have $J_1 = 0$.

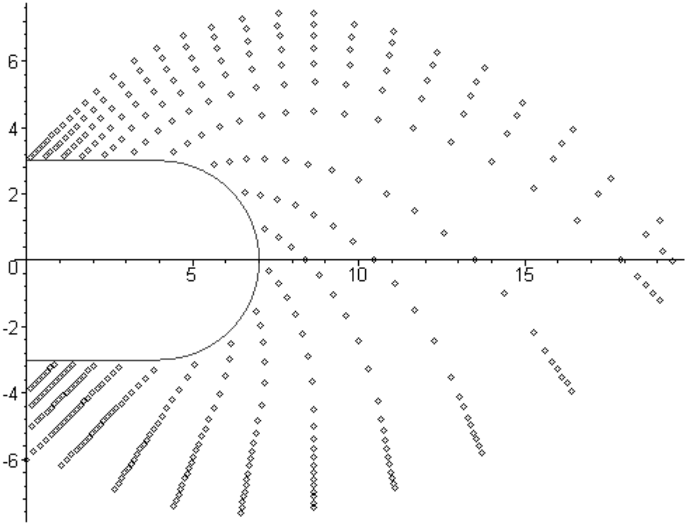

Figure: Characteristics field for the cavity

Conclusions and discussion

The relation between conservation laws of quasi-linear system and the solution of corresponding linearized system, obtained by the hodograph transformation, permits to solve the boundary value problems in the domain, where Jacobian of transformation is equal to zero.

Recently, in Ref. [2] some non-homogeneous quasilinear systems, arising in various areas of physical interest, were reduced to the homogeneous ones by using the invariance to suitable Lie groups of point transformations. As we hope, it opens the way to apply the proposed method for some non-homogeneous systems.

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