Frobenius Manifold for the Dispersionless Kadomtsev–Petviashvili Equation

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Outline

I Frobenius manifolds and systems of quasilinear PDE
II dKP equation and Vlasov equations
III Frobenius manifold for dKP
IV Principal hierarchy and solutions of dKP

Frobenius manifolds

Frobenius manifold (Dubrovin, ['92]):

- M: differentiable manifold, $dim(M) = n$,
- η: *flat, non-degenerate, (pseudo-)metric*
- $\circ$: *commutative, associative product*
- e: unity vector field: $X \circ e = X, \ \forall X$.
- E: Euler vector field

Notation:

$$c(X, Y, Z) := \eta(X, Y \circ Z)$$

$$c^i_{jk} := \eta^{is} c_{sjk}, \longrightarrow (X \circ Y)^i = X^j c^i_{jk} Y^k$$

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Frobenius manifolds

In addition, we require:

(1) Compatibility with the metric (Frobenius algebra on T_pM):

$$\eta(X, Y \circ Z) = \eta(X \circ Y, Z)$$

(2) Compatibility with the connection

$$\nabla_W c\,(X, Y, Z) = \nabla_X c\,(W, Y, Z)$$

∇: covariant derivative of η.

Remark

$(1) + (2) \implies \exists\ F \in \mathcal{C}^\infty(M)$ *s.t.*

$$c_{ijk} = \nabla_i \nabla_j \nabla_k F$$

$F =$ *potential of the Frobenius manifold* ($\Longrightarrow$ *WDVV*...)

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Moreover:

(3) Flatness of unity vector field:

$$\nabla e = 0.$$

(4) Euler vector field satisfies

$$\nabla\nabla E = 0$$
$$[E, X \circ Y] - [E, X] \circ Y - X \circ [E, Y] = X \circ Y$$
$$E(\eta(X,Y)) - \eta([E,X],Y) - \eta(X,[E,Y]) = (2-d)\,\eta(X,Y)$$

d = charge of the Frobenius manifold

i) $\exists$ a second metric: intersection form of the Frobenius manifold

$$g^{ij} = E^k \eta^{is} c^j_{ks},$$

g is flat and $\eta + \alpha g$ is flat $\forall \alpha$ (compatible flat metrics)

ii) Special coordinate sets:

a) Flat coordinates: η is constant

b) Canonical coordinates: $c^i_{jk} = \delta^i_j \delta^i_k$ (*semisimple Frob. manifold*)

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Frobenius manifolds – further properties

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Systems of quasilinear PDE

Systems of quasilinear PDEs: $u^i = u^i(x,t)$, $i = 1, \dots, n$

$$u^i_t = V^i_j(u)\, u^j_x, \qquad \lim_{|x|\mapsto\infty} u^i(x,t) = 0.$$

Hamiltonian form: (Dubrovin, Novikov ['83])

$$\{\mathcal{H}_1, \mathcal{H}_2\} = \int \frac{\delta \mathcal{H}_1}{\delta u^i(x)} \left(\eta^{ij} \frac{\partial}{\partial x} - \eta^{is} \Gamma^j_{sk} u^k_x \right) \frac{\delta \mathcal{H}_2}{\delta u^j(x)} dx, \quad \mathcal{H}_\alpha = \int H_\alpha(u) dx$$

η flat pseudo-metric, Γ Christoffel symbols

Integrability: Generalized hodograph method (Tsarev, ['94]).

$$x\,\delta^i_j + t V^i_j(u) = W^i_j(u), \qquad u^i_\tau = W^i_j(u)\, u^j_x, \quad \frac{\partial^2 u^i}{\partial t\, \partial \tau} = \frac{\partial^2 u^i}{\partial \tau\, \partial t}$$

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Principal hierarchy of a Frobenius manifold

(M, η, c, e, E) Frobenius manifold, X vector field on M

$$\eta \implies (\bar{V}_X)^i_j := \nabla_j X^i, \qquad c \implies (V_X)^i_j := c^i_{jk} X^k$$

Recursive relation. For $\alpha = 1, \ldots, n$:

$$\nabla_j X^i_{\alpha,0} = 0$$
$$\nabla_j X^i_{\alpha,m+1} = c^i_{jk} X^k_{\alpha,m}, \qquad m = 0, 1, 2, \ldots$$

Principal hierarchy of a Frobenius manifold

$$\frac{\partial u^i}{\partial t_{\alpha,m}} = (V_{\alpha,m})^i_j \frac{\partial u^j}{\partial x}, \qquad (V_{\alpha,m})^i_j = c^i_{jk} X^k_{\alpha,m},$$

The flows of the principal hierarchy pairwise commute:

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Let $f(p,x,t,y)$ be a solution of the integro - differential equations (Vlasov, or collisionless Boltzmann equation):

$$f_y = p\, f_x - A^0_x\, f_p, \tag{1}$$

$$f_t = \left(p^2 + A^0\right) f_x - \left(A^0_x p + A^1_x\right) f_p, \tag{2}$$

where

$$A^k(x,y,t) = \int_{-\infty}^{+\infty} p^k\, f(p,x,y,t)\, dp, \quad k \in \mathbb{N}.$$

The flows (1), (2) commute: $\left(f_y\right)_t = \left(f_t\right)_y$.

dKP and Vlasov equations

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dKP and Vlasov equations

Moment equations:

$$A^k_y = A^{k+1}_x + kA^{k-1}A^0_x, \quad k \in \mathbb{N} \qquad \text{(Benney chain)}$$

$$A^k_t = A^{k+2}_x + A^0 A^k_x + (k+1)A^k A^0_x + kA^{k-1}A^1_x, \qquad k \in \mathbb{N}$$

Important fact: $A^0 = \int f \, dp$ *satisfies the dKP equation:*

$$A^0_{yy} = \partial_x \left(A^0_t - A^0 A^0_x \right)$$

Indeed:

$$A^0_y = A^1_x, \qquad A^1_y = A^2_x + A^0 A^0_x, \qquad A^0_t = A^2_x + 2A^0 A^0_x.$$

Rearranging:

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dKP and Vlasov equations – Hamiltonian structure

Poisson–Vlasov bracket (Marsden, Weinstein, ['82]):

$$\{\mathcal{G},\mathcal{H}\}_{LP} := \iint f(x,p)\left\{\frac{\delta\,\mathcal{G}[f]}{\delta f(x,p)},\frac{\delta\,\mathcal{H}[f]}{\delta f(x,p)}\right\}_{x,p} dp\,dx,$$

$\{\cdot\,,\,\cdot\}_{x,p}$: canonical bracket, $\mathcal{G}$, $\mathcal{H}$ functionals of f.

Hamilton's equations:

$$\frac{\partial f}{\partial t} = \{f,\mathcal{H}\}_{LP}, \quad \Longleftrightarrow \quad \frac{\partial f}{\partial t} + \left\{f,\frac{\delta\mathcal{H}}{\delta f}\right\}_{x,p} = 0,$$

Example

$$\mathcal{H} = \frac{1}{2}\iint\left(p^2 + A^0\right)\, f\,dp\,dx \quad \Longrightarrow \quad \frac{\partial f}{\partial y} = \left\{f,\frac{p^2}{2}+A^0\right\}_{x,p}$$

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dKP – Lax formulation

Formal series

$$\lambda := p + \sum_{k=0}^{\infty} \frac{A^k}{p^{k+1}}, \qquad A^k = A^k(x + t_1, t_2, t_3, \dots).$$

The flows (dispersionless Lax equations):

$$\lambda_{t_n} = \{\Omega_n, \lambda\}_{x,p}, \quad \Omega_n = \frac{1}{n}(\lambda^n)_+, \qquad k \in \mathbb{N},$$

Zero curvature equations (Zakharov-Shabat):

$$\partial_{t_n}\Omega_m - \partial_{t_m}\Omega_n = \{\Omega_m, \Omega_n\}_{x,p}$$

Remark (bi-Hamiltonian structure)

dKP admits two compatible structures of Dubrovin–Novikov type

First metric: $\qquad \eta^{kn}(A) = (k+n)A^{k+n-1}$

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dKP and Vlasov equations

Following (Gibbons, ['81], Gibbons, Tsarev ['94,'96]): more analytical structure on λ. We consider:

$$\lambda(p) = p + \int_{-\infty}^{+\infty} \frac{f(q)}{p-q} dq, \qquad p \in \mathbb{R},$$

Asymptotic expansion

$$\lambda \sim p + \sum_{k=0}^{\infty} \frac{A^k}{p^{k+1}}, \qquad p \mapsto \infty$$

where

$$A^k = \int_{-\infty}^{+\infty} p^k f(p) dp, \qquad k \in \mathbb{N},$$

are the moments of f.

Known examples of infinite dimensional Frobenius manifolds

Krichever, Mineev-Weinstein, Wiegmann, Zabrodin, A. ['04]
Solutions of WDVV depending on infinitely many coordinates and associated to the dispersionless $2D$ Toda hierarchy. Zabrodin ['09]: dKP hierarchy

Carlet, Dubrovin, Mertens ['10]: Frobenius manifold for dispersionless $2D$ Toda hierarchy on the space of pairs of functions analytic inside and outside the unit circle respectively, and with prescribed singularities at zero and at infinity.

Wu, Xu ['11]: Frobenius manifold for the dispersionless two-component BKP hierarchy

Vlasov eqs as Hydrodynamic type systems

Following (Gibbons, A.R. ['07]): we consider Vlasov equations as 'continuous indexed'hydrodynamic type sytems:

$$f_t(p) = \int V\binom{p}{q} f_x(q)dq$$

Naive idea:

$$\{u^1, \dots, u^n\} \longrightarrow \{f(p),\, p \in \mathbb{R}\}$$

$$\{u^1(x), \dots, u^n(x)\} \longrightarrow \{f(p,x),\, p \in \mathbb{R}\}$$

$$\sum_i \longrightarrow \int dp$$

$$\frac{\partial}{\partial u^i} \longrightarrow \frac{\delta}{\delta f(p)}$$

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The Schwartz class

Analytical properties of f in the $(x, p)-$plane:

$p-$dependence: f in Schwartz class $\mathcal{S}(\mathbb{R})$ of $\mathcal{C}^\infty-$functions s.t.

$$\sup_{p\in\mathbb{R}} |\, p^k\, \partial^m f(p)\,| < +\infty, \quad k, m \in \mathbb{N}$$

$$f \in \mathcal{S}(\mathbb{R}) \quad \Longrightarrow \quad \lambda(p) = p + \int \frac{f(q)}{p-q} dq \,\in\, \mathcal{C}^\infty(\mathbb{R})$$

For simplicity: $f \in \mathcal{S}(\mathbb{R}^2)$ *in the* $(x, p)-$*plane*

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Look for linear maps

$$V\binom{p}{q} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}),$$

thus thinking at $\mathcal{S}(\mathbb{R})$ as space of tangent vectors.

Space of cotangent vectors: dual space of $\mathcal{S}(\mathbb{R})$ through the L^2−pairing

$$\langle \omega, \Phi \rangle = \int_{-\infty}^{+\infty} \omega(p)\Phi(p)\, dp, \qquad \omega \in \mathcal{S}'(\mathbb{R}),\ \Phi \in \mathcal{S}(\mathbb{R}).$$

$\mathcal{S}'(\mathbb{R})$, space of tempered distributions.

Example ($V\binom{p}{q} = p\,\delta(p-q) - f'(p)$)

$$\int V\binom{p}{q} f_x(q)dq = pf_x(p) - A^0_x f'(p) \in \mathcal{S}(\mathbb{R})$$

Later on: we will consider proper subspaces of $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$

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Example ($V\binom{p}{q} = p\,\delta(p-q) - f'(p)$)

$$\int V\binom{p}{q} f_x(q) dq = pf_x(p) - A_x^0 f'(p) \in \mathcal{S}(\mathbb{R})$$

Later on: we will consider proper subspaces of $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$

First contravariant metric: $\eta^{(p,q)}$

Proposition (Gibbons, A.R., ['07])

The LP-bracket can be written as a continuous index bracket of hydrodynamic type

$$\{\mathcal{G},\mathcal{H}\}=\iiint\frac{\delta\,\mathcal{G}[f]}{\delta f(x,p)}\left(\eta^{(p,q)}\frac{\partial}{\partial x}+\int\Gamma\binom{p,q}{r}f_x(r)dr\right)\frac{\delta\,\mathcal{H}[f]}{\delta f(x,p)}dp\,dq\,dx,$$

Continuous form of the first metric

$$\eta^{(p\,q)}[f]=-f'(p)\delta(p-q)\quad\Longrightarrow\quad\eta^{nm}[A]=(n+m)A^{n+m-1}$$

Remark

η is not invertible on $\mathcal{S}'(\mathbb{R})$. We consider $\mathcal{O}_M(\mathbb{R})\subset\mathcal{S}'(\mathbb{R})$, the space of $\mathcal{C}^\infty(\mathbb{R})$ functions which, together with all their derivatives, grow at infinity slower than some polynomial.

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First covariant metric: $\eta_{(p,q)}$

Denote $\mathcal{V} \subset \mathcal{S}(\mathbb{R})$ the linear space of functions of the form

$$X^{(p)} = h(p) f'(p), \qquad h \in \mathcal{O}_M(\mathbb{R})$$

On $\mathcal{V}$, the inverse metric of η:

$$\eta_{(p\,q)} = -\frac{1}{f'(p)}\delta(p-q),$$

is well defined. Action on vectors X_1, $X_2 \in \mathcal{V}$:

$$\eta(X_1, X_2) = \iint X_1^{(p)} \eta_{(p,q)} X_2^{(q)}\, dp\, dq = -\int \frac{X_1^{(p)} X_2^{(p)}}{f'(p)} dp.$$

The metric is flat: the (continuous) Riemann tensor is zero

Christoffel symbols

$$\Gamma\binom{p}{q\,r} = \frac{1}{f'(q)}\delta'(q-p)\delta(q-r).$$

Covariant derivative, acting on $X \in \mathcal{V}$ as

$$\nabla_{(q)}X^{(p)} = \frac{\delta X^{(p)}}{\delta f(q)} + \int \Gamma\binom{p}{q\,r} X^{(r)} dr = \frac{\delta X^{(p)}}{\delta f(q)} + \delta'(q-p)X^{(p)},$$

and on $h \in \mathcal{O}_M(\mathbb{R})$ as

$$\nabla_{(p)}h_{(q)} = \frac{\delta h_{(q)}}{\delta f(p)} - \int \Gamma\binom{r}{p\,q} h_{(r)} dr = \frac{\delta h_{(q)}}{\delta f(p)} - \delta(p-q)\frac{\partial h_{(p)}}{\partial p}.$$

The two actions can be proved to be consistent, and can be extended to higher order multilinear maps.

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Structure constants of the algebra

Look for a bilinear map $\circ : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$, of the form

$$(X \circ Y)^{(p)} = \iint X^{(q)} c\binom{p}{qs} Y^{(s)} dqds, \qquad X, Y \in \mathcal{V}$$

Consider the quantities

$$c\begin{pmatrix} p \\ q\,r \end{pmatrix} = \frac{\delta(p-r)}{p-q} - \frac{f'(p)}{f'(q)} \frac{\delta(q-r)}{p-q} + \frac{\delta(p-q)}{q-r} - \frac{\lambda'(p)}{f'(p)} \delta(q-r)\delta(q-p),$$

where

$$\lambda'(p) = 1 + \fint_{-\infty}^{+\infty} \frac{f'(q)}{p-q} dq.$$

The space $\mathcal{V}$ is closed under the product $\circ$. Moreover, we have:

Structure constants of the algebra – properties

- Symmetry

$$c\binom{p}{q\,r} = c\binom{p}{r\,q}.$$

- Associativity

$$\int c\binom{p}{q\,s} c\binom{s}{l\,r} ds = \int c\binom{p}{l\,s} c\binom{s}{q\,r} ds.$$

- Compatibility with the metric

$$\int \eta_{(p\,q)} c\binom{q}{r\,s} dq = \int \eta_{(r\,q)} c\binom{q}{p\,s} dq.$$

- Compatibility with the connection

$$\nabla_{(l)} c\binom{p}{q\,r} = \nabla_{(q)} c\binom{p}{l\,r}.$$

The unity vector: e

Proposition

The element $e^{(p)} = -f'(p) \in \mathcal{V}$ is the unity vector field.

Indeed:

$$(X \circ e)^{(p)} = \iint X^{(q)} c\binom{p}{qr} e^{(r)} dq\, dr = \int X^{(q)} \delta(p-q) dq = X^{(p)}$$

Moreover,

$$\nabla_{(q)} e^{(p)} = \frac{\delta\, e^{(p)}}{\delta f(q)} + \int \Gamma\binom{p}{q\,s} e^{(s)} ds = -\delta'(p-q) - \delta'(q-p) = 0,$$

$\Longrightarrow$ e is flat.

Euler vector field: E

Proposition

The element

$$E^{(p)} = f(p) - p\,f'(p) \ \in \mathcal{S}(\mathbb{R}) \setminus \mathcal{V}$$

satisfies

$$\nabla_p \nabla_q E^{(r)} = 0,$$

$$\int E^{(s)} \frac{\delta c\binom{p}{qr}}{\delta f(s)}\, ds - \int c\binom{s}{qr} \frac{\delta E^{(p)}}{\delta f(s)}\, ds + \int \frac{\delta E^{(s)}}{\delta f(q)} c\binom{p}{sr}\, ds + \int \frac{\delta E^{(s)}}{\delta f(r)} c\binom{p}{qs}\, ds = c\binom{p}{qr}$$

$$\int E^{(r)} \frac{\delta \eta_{(p\,q)}}{\delta f(r)}\, dr + \int \eta_{(r\,q)} \frac{\delta E^{(r)}}{\delta f(p)}\, dr + \int \eta_{(p\,q)} \frac{\delta E^{(r)}}{\delta f(q)}\, dr = 3\,\eta_{(p\,q)},$$

and it is therefore the Euler vector field of the Frobenius manifold. Moreover, the Frobenius manifold has charge $d = -1$.

Potential of the Frobenius manifold

Theorem

The functional (logarithmic energy with external field)

$$F = \frac{1}{2}\iint \log|p-q| f(p) f(q) dp\, dq + \frac{1}{2}\int p^2 f(p) dp.$$

satisfies the condition

$$\nabla_{(p)}\nabla_{(q)}\nabla_{(r)} F = c_{(p\,q\,r)}$$

and is therefore the potential of the Frobenius manifold.

Random Matrix Theory: equilibrium measure for the large N limit of the partition function for the Gaussian Unitary Ensemble. (Wiegmann, Zabrodin ['00], Elbau, Felder ['05],...)

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PDE associated with the product c: dKP hierarchy

Example ($X_2^{(p)} = -p\,f'(p) \in \mathcal{V}$)

$$
\begin{aligned}
V_{X_2}\binom{p}{q} &:= \int c\binom{p}{q\ r} X_2^{(r)} dr \\
&= -\frac{pf'(p)}{p-q} + \frac{qf'(p)}{p-q} - \delta(p-q)\left(\int \frac{rf'(r)}{p-r} dr - p\lambda'(p)\right) \\
&= -f'(p) - \delta(p-q)\left(\int \frac{(r-p)f'(r)}{p-r} dr - p\right) \\
&= p\,\delta(p-q) - f'(p),
\end{aligned}
$$

Example ($X_3^{(p)} = -\left(p^2 + 2A^0\right) f'(p) \in \mathcal{V}$)

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V_{X_3}\binom{p}{q} = \left(p^2 + A^0\right)\delta(p-q) - (p+q)f'(p)
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$$= -f'(p) - \delta(p-q)\left(\int \frac{(r-p)f'(r)}{p-r} dr - p\right)$$

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Principal hierarchy – Flat vector fields

The functionals

$$\mathcal{H}_{h,0} = \int H_{h,0}\, dx, \qquad H_{h,0} = \int h(f)\, dp,$$

are Casimirs of the Lie–Poisson bracket. Consider the vector fields

$$X^{(p)}_{h,0} = \int \frac{\delta H_{h,0}}{\delta f(q)}\, \eta^{(q,p)} dq = -h'(f(p))f'(p) \in \mathcal{V}, \qquad (3)$$

Proposition

The vector fields (3) are flat:

$$\nabla_{(q)} X^{(p)}_{h,0} = 0.$$

For $h(f) = f$ we get the unity $e^{(p)} = -f'(p)$.

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Principal Hierarchy – Primary flows

Primary flows of the principal hierarchy:

$$\partial_{t_{h,0}} f(p) = \int V_{h,0}\binom{p}{q} f_x(q) dq, \qquad V_{h,0}\binom{p}{q} := \int c\binom{p}{qr} X^{(r)}_{h,0} \quad (4)$$

Proposition

The flows (4) are Hamiltonian of the form

$$\partial_{t_{h,0}} f(p) = \left\{ f(p), \int H_{h,1}\, dx \right\}_{LP},$$

with the Poisson–Vlasov bracket, and the Hamiltonian

$$H_{h,1} = \int h(f(p))\, \lambda(p)\, dp.$$

Theorem (Principal Hierarchy for dKP)

The vector fields

$$X^{(p)}_{h,n} := -\frac{\delta H_{h,n}}{\delta f(p)}\, f'(p), \quad H_{h,n} = \frac{1}{n!}\int h(f(p))\lambda(p)^n\, dp,$$

$n \in \mathbb{N}$, *satisfy the recurrence relations*

$$\nabla_{(q)} X^{(p)}_{h,n+1} = \int c\binom{p}{qr} X^{(r)}_{h,n},$$

of the principal hierarchy. The corresponding commuting flows

$$\partial_{t_{h,n}} f(p) = \int V_{h,n}\binom{p}{q} f_x(q) dq, \qquad V_{h,n}\binom{p}{q} := \int c\binom{p}{qr} X^{(r)}_{h,n},$$

are Hamiltonian

$$\partial_{t_{h,n}} f(p) = \{f, \mathcal{H}_{h,n+1}\}_{LP}, \quad \mathcal{H}_{h,n+1} = \int H_{h,n+1} dx.$$

The generic conserved quantity

Proposition

Let $k(\mu, \nu)$, be a function of two variables, s.t. the integral

$$\mathcal{K} = \iint k(f(p,x), \lambda(p,x)) dp\, dx,$$

converges. Then, $\mathcal{K}$ is a conserved quantity for the dKP hierarchy.

Direct result: no dependence on the Frobenius manifold!!

$$\text{Principal hierarchy} \Longrightarrow h(f, \lambda) = \frac{1}{n!} h(f)\ \lambda^n$$

$$\text{Classical Hamiltonians} \Longrightarrow h(f, \lambda) = \frac{1}{n!} f\ \lambda^n + C(f)$$

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Hodograph formula for dKP

Analogy with finite dimensional case:

$$x\,V_{X_1}\binom{p}{q} + y\,V_{X_2}\binom{p}{q} + t\,V_{X_3}\binom{p}{q} - V_{X_K}\binom{p}{q} = 0,$$

with $X_1 = -f'(p), \quad X_2 = -pf'(p), \quad X_3 = -(p^2 + 2A^0)f'(p)$

$$K = \int k(f(p), \lambda(p))dp \quad \Longrightarrow \quad X_K = -\frac{\delta K}{\delta f(r)} f'(r)$$

$$\int c\binom{p}{q\,r}\left(x + y\,r + t\,(r^2 + 2A^0) - \frac{\delta K}{\delta f(r)}\right) f'(r)dr = 0,$$

Nonlinear integral equation (also in Manakov, Santini ['06]):

$$x + y\,p + t\,(p^2 + 2A^0) = \partial_1 k(f(p), \lambda(p)) - \int \frac{\partial_2 k(f(q), \lambda(q))}{p - q} dq$$

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Hodograph formula = stationary point of a dKP conserved quantity

Indeed: the functional

$$\mathcal{H}_K = \int \left(x\, A^0 + y\, A^1 + t \left(A^2 + (A^0)^2 \right) - K \right) dx.$$

is a constant of motion for the y and t flows of the hierarchy, and

$$\text{Hodograph formula} \iff \frac{\delta\, \mathcal{H}_K}{\delta f(p,x)} = 0.$$

Moreover:

f satisfies Hodograph $\implies$ $A^0 = \int f\, dp$ is a solution of dKP

Hodograph formula – Solutions of dKP

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J. Gibbons, A. R.: Differential geometry of Hydrodynamic Vlasov equations *J. Geom. Phys.*, 57(9): 1815–1828, (2007)

A. R.: Frobenius manifold for the dispersionless Kadomtsev – Petviashvili equation arxiv.org/abs/1008.2128, (2010)