Whitham Theory for Perturbed Integrable Equations and its Applications

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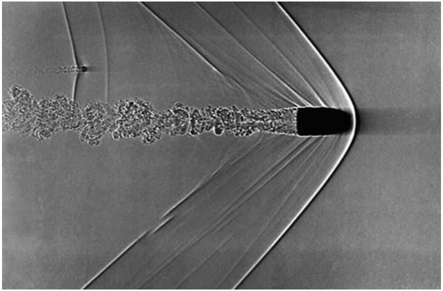
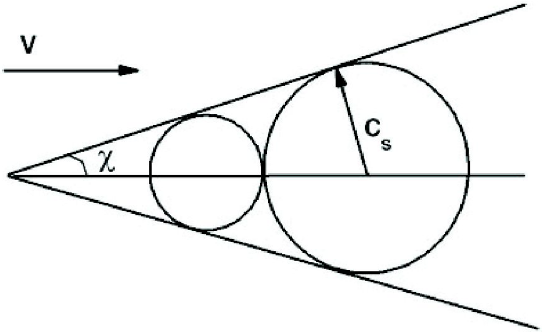
Outline

E. Mach experiment



Nuclear test: July 1945, New Mexico

Mach-Cherenkov cone

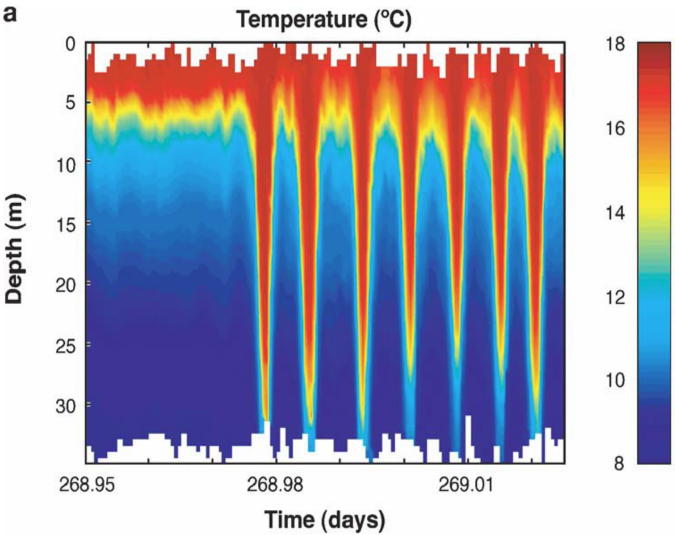


$$\sin\chi = \frac{c_s}{V} = \frac{1}{M}$$

Undular bore: Dordogne river, France

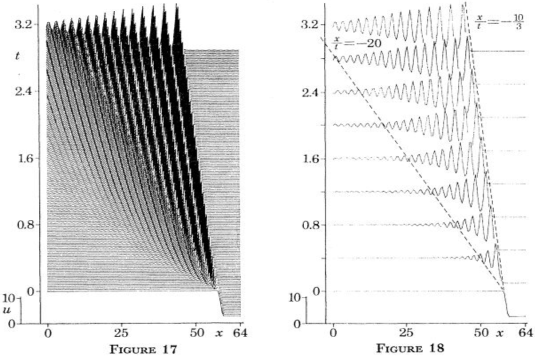
Morning glory: Australia

Internal water wave: Oregon coast



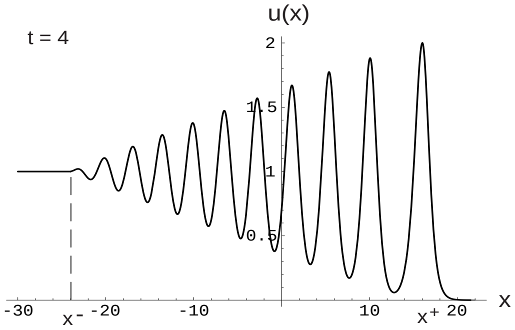
Numerical solution of the KdV equation $u_t + uu_x + u_{xxx} = 0$ with a step-like initial condition

$$u(x,0) = \begin{cases} 0, & x < 0 \\ -1, & x > 0 \end{cases}$$



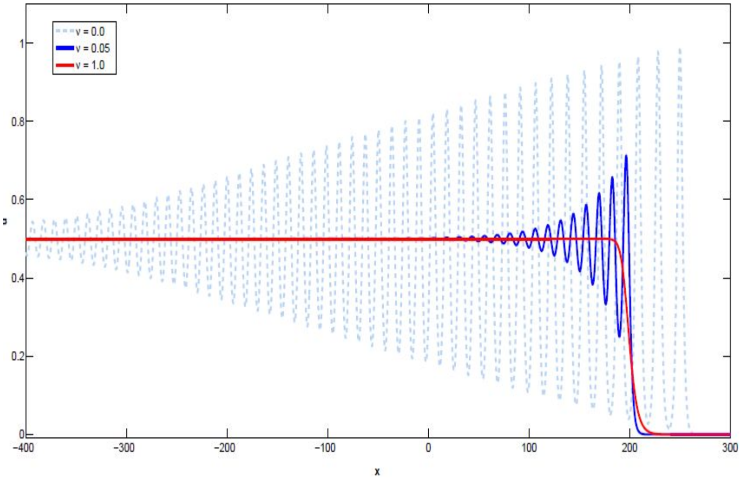
From *B. Fornberg and G.B. Whitham, Phil. Trans. Royal Soc. London,* **289,** 32 (1978)

According to Gurevich-Pitaevskii (1973) approach based on the Whitham (1965) modulation theory theory and in agreement with numerics, if the jump of a step-like pulse is equal to 1, then the amplitude of a leading soliton is equal to 2.



What happens if a small friction is added?

$$u_t + 6uu_x + u_{xxx} = \nu u_{xx}$$



In the papers

R.S. Johnson, J. Fluid Mech. **42,** 49 (1970)

V.V. Avilov, I.M. Krichever, S.P. Novikov, DAN SSSR, **294,** 325 (1987)

A.V. Gurevich, L.P. Pitaevskii, ZhETF **93,** 871 (1987)

it was shown that asymptotically, instead of an expanding shock, we get a stationary shock with the amplitude of a leading soliton equal to 3/2.

Thus even small disturbances of the evolutional equations lead to qualitative changes in the asymptotic evolution. Hence we need the perturbation theory for Whitham modulation approach.

Whitham theory

We suppose that non-perturbed evolution equations are completely integrable in framework of Ablowitz-Kaup-Newell-Segur (AKNS) scheme.

$$u_{m,t} = K_m(u_n, u_{n,x}, \dots) + R_m(x, t, u_n, u_{n,x}, \dots), \qquad m, n = 1, \dots, N,$$

This means that the undisturbed equations $u_{m,t} = K_m(u_n, u_{n,x}, \dots)$ can be expressed as compatibility conditions of two linear equations

$$\psi_{xx} = \mathcal{A}\psi, \quad \psi_t = -\frac{1}{2}\mathcal{B}_x\psi + \mathcal{B}\psi_x$$

where $\mathcal{A} = \mathcal{A}(u_n, u_{n,x}, \dots; \lambda)$, $\mathcal{B} = \mathcal{B}(u_n, u_{n,x}, \dots; \lambda)$, and λ is a free spectral parameter.

Example 1 (Gardner, Green, Kruskal, Miura, (1967)) The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

is a consequence of the compatibility condition $(\psi_t)_{xx} = (\psi_{xx})_t$ with

$$\mathcal{A} = -(u+\lambda), \qquad \mathcal{B} = 4\lambda - 2u.$$

Example 2 (Zakharov, Shabat, (1971)). The NLS equation

$$\mathrm{i}u_t + u_{xx} + 2|u|^2 u = 0$$

corresponds to

$$\mathcal{A} = -\left(\lambda - \frac{\mathrm{i}u_x}{2u}\right)^2 - |u|^2 - \left(\frac{u_x}{2u}\right)_x, \qquad \mathcal{B} = 2\lambda + \frac{\mathrm{i}u_x}{u}.$$

Finite-gap integration method

(S.P. Novikov, B.A. Dubrovin, I.M. Krichever et al., 1974–1976)

The periodic solution

$$u_m(x,t) = u_m(x,t;\lambda_1,\ldots,\lambda_M), \quad m = 1,\ldots,N.$$

of unperturbed equations

$$u_{m,t} = K_m(u_n, u_{n,x},\ldots), \qquad m,n = 1,\ldots,N,$$

depends on space coordinate x and time t and several parameters $\lambda_1,\ldots,\lambda_M$ ("integration constants") which appear naturally in the "finite-gap integration method" as zeroes of some polynomial

$$P(\lambda) = \prod_{i=1}^{M}(\lambda - \lambda_i).$$

The second-order differential equation

$$\psi_{xx} = \mathcal{A}\psi, \quad \mathcal{A} = \mathcal{A}(u_n, u_{n,x}, \ldots; \lambda)$$

has two basis solutions ψ^+, ψ^-. Their product

$$g = \psi^+\psi^-$$

satisfies the 3rd order differential equation

$$g_{xxx} - 2\mathcal{A}_x g - 4\mathcal{A} g_x = 0,$$

which upon multiplication by $g/2$ can be integrated once to give

$$\frac{1}{2} g g_{xx} - \frac{1}{4} g_x^2 - \mathcal{A} g^2 = \pm P(\lambda).$$

Quasiperiodic solutions correspond to polynomial dependence of $P(\lambda)$ on λ.

From

$$\psi_t = -\frac{1}{2}\mathcal{B}_x\psi + \mathcal{B}\psi_x, \quad \mathcal{B} = \mathcal{B}(u_n, u_{n,x}, \ldots; \lambda)$$

we get the time-dependence of $g = \psi^+\psi^-$,

$$g_t = \mathcal{B}g_x - \mathcal{B}_x g,$$

which yields the generation function of conservation laws:

$$\left(\frac{1}{\tilde{g}}\right)_t = \left(\frac{\mathcal{B}}{\tilde{g}}\right)_x$$

where

$$\tilde{g} = \frac{\lambda^{r/2}}{\sqrt{P(\lambda)}}g, \quad \tilde{g}|_{\lambda\to\infty} = 1.$$

We expand $\tilde{g}$ and $1/\tilde{g}$ in inverse powers of λ,

$$\tilde{g} = \sum_{n=0}^{\infty} \frac{g_n}{\lambda^n}, \quad \frac{1}{\tilde{g}} = \sum_{n=0}^{\infty} \frac{g_{-n}}{\lambda^n}, \quad g_0 = 1,$$

and the coefficients g_n, g_{-n} are calculated in recurrent way. These coefficients g_{-n} serve as the densities of the conservation laws and the coefficients of similar expansion of $\mathcal{B}/\tilde{g}$ in inverse powers of λ serve as the corresponding fluxes,

$$(g_{-n})_t = [(\mathcal{B}/\tilde{g})_{-n}]_x \, ,$$

where $(\mathcal{B}/\tilde{g})_{-n}$ denotes the coefficients in the expansion

$$\frac{\mathcal{B}}{\tilde{g}} = \sum_{n=0}^{\infty} \frac{(\mathcal{B}/\tilde{g})_{-n}}{\lambda^n}.$$

$$\left(\frac{\sqrt{P(\lambda)}}{\lambda^{r/2}}\cdot\frac{1}{g}\right)_t-\left(\frac{\sqrt{P(\lambda)}}{\lambda^{r/2}}\cdot\frac{\mathcal{B}}{g}\right)_x=0.$$

Here the zeroes λ_i of the polynomial $P(\lambda)$ are slow functions of x and t which have to be differentiated with respect to x and t. Then we obtain the terms with

$$\frac{1}{\sqrt{\lambda-\lambda_i}}\frac{\partial\lambda_i}{\partial t}\quad\text{and}\quad\frac{1}{\sqrt{\lambda-\lambda_i}}\frac{\partial\lambda_i}{\partial x},$$

which are singular at $\lambda\to\lambda_i$. Hence we obtain

$$\frac{\partial\lambda_i}{\partial t}+v_i\frac{\partial\lambda_i}{\partial x}=0\quad i=1,\dots,M,$$

where

$$v_i=-\frac{\langle\mathcal{B}/g\rangle}{\langle 1/g\rangle},\quad i=1,\dots,M,\quad \langle\mathcal{F}\rangle=\frac{1}{L}\int_0^L\mathcal{F}dx.$$

Example. The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

corresponds to

$$\mathcal{A} = -(u+\lambda), \quad \mathcal{B} = 4\lambda - 2u.$$

One-phase solution is obtained from

$$\tfrac{1}{2}gg_{xx} - \tfrac{1}{4}g_x^2 + (u+\lambda)g^2 = P(\lambda)$$

if

$$P(\lambda) = \prod_{i=1}^{3}(\lambda - \lambda_i) = \lambda^3 - s_1\lambda^2 + s_2\lambda - s_3, \quad g = \lambda - \mu$$

and is given by

$$u(x,t) = \lambda_3 - \lambda_1 - \lambda_2 - 2(\lambda_3 - \lambda_2)\mathrm{sn}^2(\sqrt{\lambda_3 - \lambda_1}\,(x - Vt), m),$$

$$L = \frac{1}{2}\oint \frac{\mathrm{d}\mu}{\sqrt{-P(\mu)}} = \frac{2\mathrm{K}(m)}{\sqrt{\lambda_3 - \lambda_1}}, \quad V = -2\sum \lambda_i, \quad m = \frac{\lambda_3 - \lambda_2}{\lambda_3 - \lambda_1}.$$

Whitham equations for KdV cnoidal wave

In Whitham modulation theory the parameters λ_i become slow functions of x and t which change little in one wavelength L and one period. Evolution of the parameters λ_i is governed by the Whitham equations

$$\frac{\partial \lambda_i}{\partial t} + v_i \frac{\partial \lambda_i}{\partial x} = 0 \quad i = 1, 2, 3,$$

where $v_i(\lambda)$ are Whitham velocities

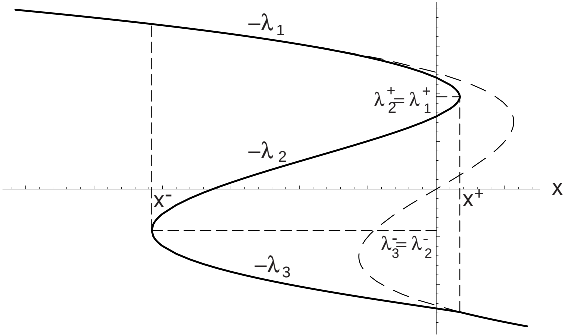
$$v_i = -\frac{\langle \mathcal{B}/g \rangle}{\langle 1/g \rangle} = \frac{\langle (4\lambda_i - 2u)/g \rangle}{\langle 1/g \rangle}, \quad i = 1, 2, 3.$$

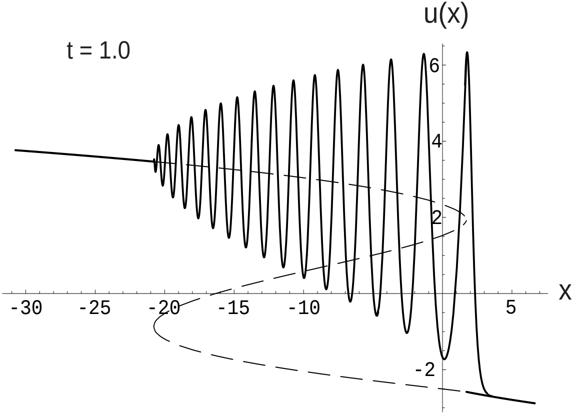
and

$$v_i = \left(1 - \frac{L}{\partial_i L}\partial_i\right) V, \quad \partial_i \equiv \frac{\partial}{\partial \lambda_i}, \quad i = 1, 2.3, \quad V = -2\sum \lambda_i.$$

Solutions $\lambda_i = \lambda_i(x, t)$ of the Whitham equations substituted into the cnoidal wave expression, $u = u(x, t; \lambda_1(x, t), \lambda_2(x, t), \lambda_3(x, t))$ describe the dispersive shock.

Dispersive KdV shock generated after wave breaking point




u(x)
t = 1.0
6
4
2
-2
x
-30
-25
-20
-15
-10
5

Perturbed Whitham theory

(A.M. Kamchatnov, Physica D, 188, 247 (2004))

If we take into account the perturbation terms in the evolution equations

$$u_{m,t} = K_m(u_n, u_{n,x}, \ldots) + R_m(x, t, u_n, u_{n,x}, \ldots), \qquad m, n = 1, \ldots, N,$$

then the parameters λ_i evolve not only because of modulation of the wave, but also due to contribution of perturbation.

Example: perturbed KdV equation

$$u_t = -6uu_x - u_{xxx} + R(u, u_x, u_{xx}, \ldots)$$

We have to calculate modification of conservation laws due to disturbance.

Perturbed Whitham theory

Example: perturbed KdV equation

$$u_t = -6uu_x - u_{xxx} + R(u, u_x, u_{xx}, \dots)$$

Conservation laws are modified to

$$\frac{\partial g_{-n}}{\partial t} = \sum_{k=0}^{l} \frac{\partial g_{-n}}{\partial u^{(k)}} \frac{\partial u^{(k)}}{\partial t} = \sum_{k=0}^{l} \frac{\partial g_{-n}}{\partial u^{(k)}} \frac{\partial^k}{\partial x^k} u_t$$

$$= [(\mathcal{B}/\tilde{g})_{-n}]_x + \sum_{k=0}^{l} \frac{\partial g_{-n}}{\partial u^{(k)}} \frac{\partial^k R}{\partial x^k}.$$

or

$$\left(\frac{1}{\tilde{g}}\right)_t - \left(\frac{\mathcal{B}}{\tilde{g}}\right)_x = \sum_{n=0}^{\infty} \frac{1}{\lambda^n} \left(\frac{\widehat{\delta}}{\delta u} g_{-n}\right) R = \left(\frac{\widehat{\delta}}{\delta u} \frac{1}{\tilde{g}}\right) R.$$

Dubrovin's lemma (1975):

$$\tilde{g} = 2\lambda \frac{\widehat{\delta}}{\delta u} \frac{1}{\tilde{g}}.$$

We get

$$\left(\frac{\sqrt{P(\lambda)}}{\lambda^{1/2}} \cdot \frac{1}{g} \right)_t - \left(\frac{\sqrt{P(\lambda)}}{\lambda^{1/2}} \cdot \frac{4\lambda - 2u}{g} \right)_x = \frac{1}{2\lambda} \sqrt{\frac{\lambda}{P(\lambda)}} \cdot gR.$$

and the Whitham equations take the form

$$\frac{\partial \lambda_i}{\partial t} + v_i \frac{\partial \lambda_i}{\partial x} = -\frac{1}{\prod_{j \neq i}(\lambda_i - \lambda_j)} \cdot \frac{\langle gR \rangle}{\langle 1/g \rangle} \quad i = 1, 2, 3.$$

Generalization to the AKNS scheme

We arrive at modified Whitham equations

$$\frac{\partial \lambda_i}{\partial t} + v_i \frac{\partial \lambda_i}{\partial x} = \frac{1}{\prod_{j\neq i}(\lambda_i - \lambda_j)} \cdot \frac{\pm 1}{\langle 1/g \rangle} \sum_{m=1}^{N} \sum_{l=0}^{A_m} \Big\langle \frac{\partial \mathcal{A}}{\partial u_m^{(l)}} \frac{\partial^l R_m}{\partial x^l} g \Big\rangle,$$

where $v_i(\lambda)$ are usual Whitham velocities, the angle brackets denote averaging over wavelength,

$$\langle \mathfrak{F} \rangle = \frac{1}{L} \int_0^L \mathfrak{F} dx$$

and everywhere λ is put equal to λ_i.

Kaup-Boussinesq-Burgers equation

G.A. El, R.H.J. Grimshaw, AMK, Chaos, **15,** 037102 (2005)

Two-directional shallow flow can be described by the Kaup-Boussinesq system

$$h_t + (hu)_x + \tfrac{1}{4}u_{xxx} = 0,$$
$$u_t + uu_x + h_x = \nu u_{xx}.$$

The unperturbed system ($\nu = 0$) is completely integrable and corresponds to

$$\mathcal{A} = \left(\lambda - \frac{1}{2}u\right)^2 - h, \qquad \mathcal{B} = -\left(\lambda + \frac{1}{2}u\right).$$

Its periodic solution is parameterized by the zeroes of the fourth-degree polynomial

$$P(\lambda) = \prod_{i=1}^{4}(\lambda - \lambda_i) = \lambda^4 - s_1\lambda^3 + s_2\lambda^2 - s_3\lambda + s_4$$

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4.$$

The periodic solution has the form

$$u(x,t) = s_1 - 2\mu(\theta), \quad h(x,t) = \frac{1}{4}s_1^2 - s_2 - 2\mu^2(\theta) + s_1\mu(\theta),$$

where

$$\mu(\theta) = \frac{\lambda_2(\lambda_3-\lambda_1) - \lambda_1(\lambda_3-\lambda_2)\mathrm{sn}^2\left(\sqrt{(\lambda_4-\lambda_2)(\lambda_3-\lambda_1)}\,\theta, m\right)}{\lambda_3-\lambda_1-(\lambda_3-\lambda_2)\mathrm{sn}^2\left(\sqrt{(\lambda_4-\lambda_2)(\lambda_3-\lambda_1)}\,\theta, m\right)}.$$

$$m = \frac{(\lambda_3-\lambda_2)(\lambda_4-\lambda_1)}{(\lambda_4-\lambda_2)(\lambda_3-\lambda_1)}.$$

and $\theta = x - \frac{1}{2}s_1 t$.

In a modulated wave and with account of friction the parameters λ_i satisfy the Whitham equations

$$\frac{\partial \lambda_i}{\partial t} + v_i \frac{\partial \lambda_i}{\partial x} = \rho_i, \quad i = 1, 2, 3, 4.$$

where

$$v_i = \frac{s_1}{2} - \frac{L}{2}\left(\frac{\partial L}{\partial \lambda_i}\right)^{-1},$$

$$\rho_i = \frac{1}{\prod_{j \neq i}(\lambda_i - \lambda_j)} \cdot \frac{8\nu}{(\partial L / \partial \lambda_i)} \int_{\lambda_2}^{\lambda_3} (\mu - s_1/4)\sqrt{P(\mu)}\, d\mu,$$

$$L = \int_{\lambda_2}^{\lambda_3} \frac{d\mu}{\sqrt{P(\mu)}} = \frac{2\mathrm{K}(m)}{\sqrt{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)}}.$$

Steady solution of the Whitham equations

If we look for the solution of the Whitham equations in the form

$$\lambda_i = \lambda_i(\theta), \qquad \theta = x - ct, \quad c = s_1/2,$$

then we can reduce it to the system of ordinary differential equations

$$\frac{d\lambda_i}{d\theta} = \frac{Q}{\prod_{j\neq i}(\lambda_i - \lambda_j)},$$

where the factor

$$Q = -\frac{8\nu}{L}\int_{\lambda_2}^{\lambda_3}(\mu - s_1/4)\sqrt{P(\mu)}\,d\mu$$

is the same for all $i = 1, 2, 3, 4$.

Jacobi identities (Dissertatio 1825; Werke Bd. 3):

$$\sum_{i=1}^{n}\frac{1}{\prod_{j\neq i}(\lambda_i-\lambda_j)}=0,\quad \sum_{i=1}^{n}\frac{\sum_j'\lambda_j}{\prod_{j\neq i}(\lambda_i-\lambda_j)}=0,$$

$$\sum_{i=1}^{n}\frac{\sum_{j,k}'\lambda_j\lambda_k}{\prod_{j\neq i}(\lambda_i-\lambda_j)}=0,\ldots\quad \sum_{i=1}^{n}\frac{1}{\lambda_i\prod_{j\neq i}(\lambda_i-\lambda_j)}=\frac{(-1)^{n-1}}{s_n}$$

where prime means that all terms with the factor λ_i are omitted in the corresponding sum. The special structure of Whitham equations provides three integrals $s_1,\ s_2,\ s_3$,

$$\frac{ds_1}{dx}=0,\quad \frac{ds_2}{dx}=0,\quad \frac{ds_3}{dx}=0.$$

Thus, in the steady solution only the last coefficient s_4 varies with $\theta=x-ct$ according to the equation

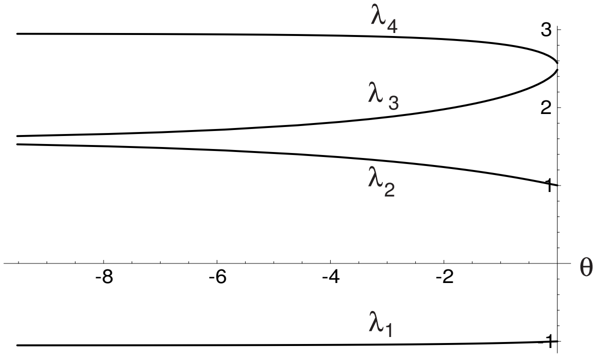
$$\frac{ds_4}{d\theta}=\frac{8\nu}{L}\int_{\lambda_2(s_4)}^{\lambda_3(s_4)}(\mu-s_1/4)\sqrt{P(\mu)}\,d\mu,$$

which can be easily solved numerically for given constant values of $s_1,\ s_2,\ s_3$.

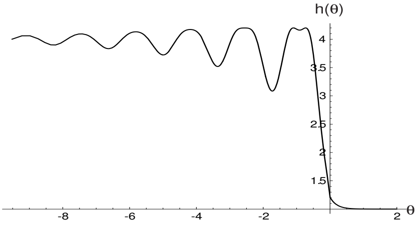
Equation

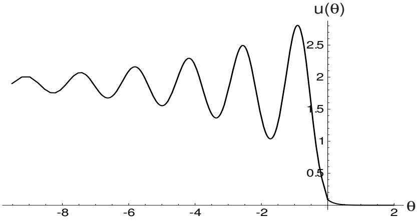
$$P(\lambda) = \lambda^4 - s_1\lambda^3 + s_2\lambda^2 - s_3\lambda + s_4 = 0$$

yields the zeroes λ_i as functions of $s_4 = s_4(\theta)$:

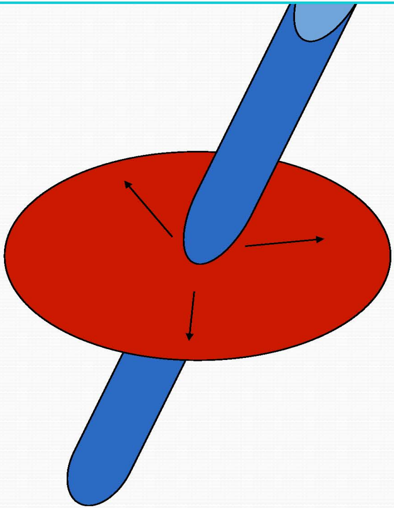


Substitution of $\lambda_i(\theta)$ gives the oscillating structure of the undular bore

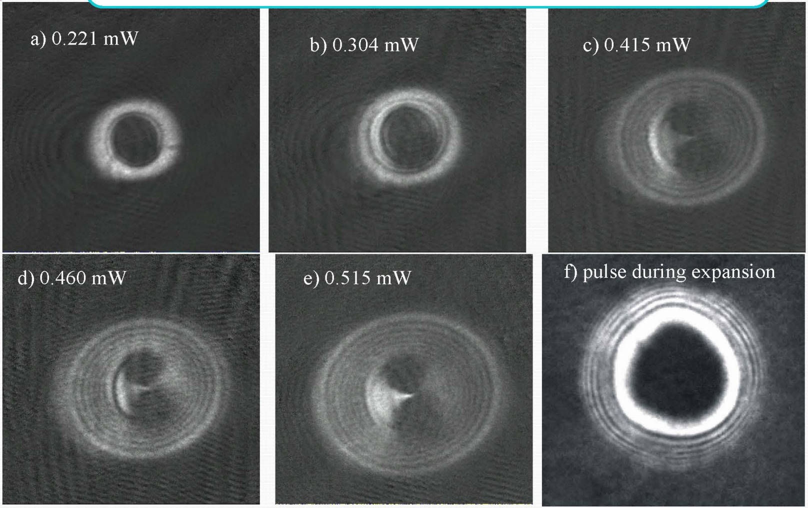




Blast waves created by a laser beam (E.Cornell, 2005; M.A. Hoefer et al, 2006)

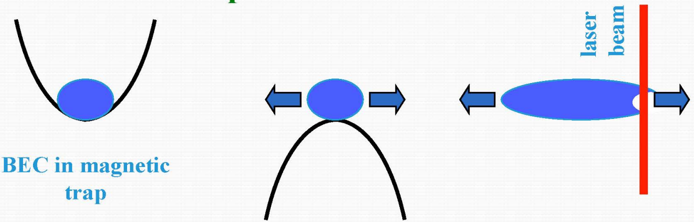


Experimental pictures of dispersive shock waves in BEC (E.Cornell 2005)



Dispersive shocks created by a flow of BEC past an obstacle (E. Cornell, 2005)

How to create a supersonic flow of BEC?

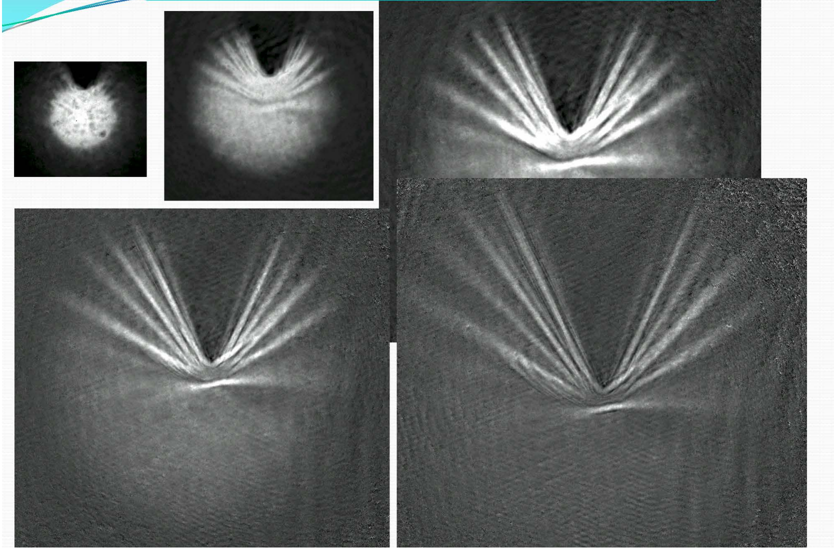


BEC in magnetic trap

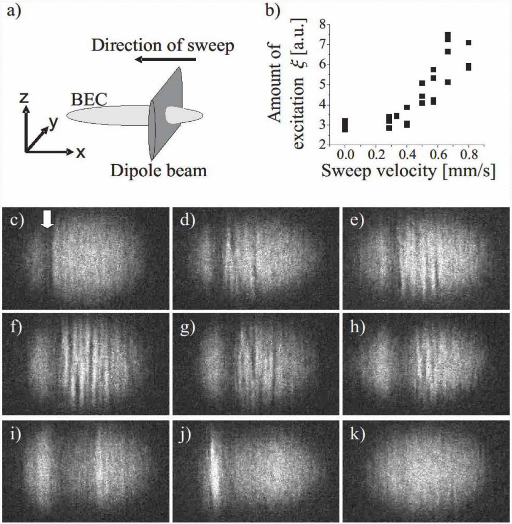
Inversion of the trapping potential

BEC expands past an obstacle created by a laser beam

Experiment (E.Cornell, 2005)



Engels-Atherton experiment with penetrable barrier (PRL, 2007)



EQUATIONS OF BEC DYNAMICS

Gross-Pitaevskii equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(\mathbf{r})\psi + g|\psi|^2\psi,$$

$$\psi(\mathbf{r}, t) = \sqrt{n(\mathbf{r}, t)}\exp\left(\frac{i}{\hbar}\int^{\mathbf{r}}\mathbf{u}(\mathbf{r}', t)d\mathbf{r}'\right)$$

$n(r,t) = |\psi|^2$ - BEC density

$u(r,t)$ - flow velocity

$$n_t + \nabla(n\mathbf{u}) = 0,$$

$$\mathbf{u}_t + (\mathbf{u}\nabla)\mathbf{u} + \frac{g}{m}\nabla n + \frac{\hbar^2}{2m^2}\nabla\left(\frac{(\nabla n)^2}{4n^2} - \frac{\triangle n}{2n}\right) = -\nabla V$$

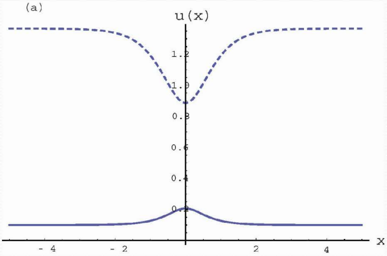
Flow past a penetrable barrier (Leszczyszyn, El, Gladush, Kamchatnov, 2009)

Hydraulic approximation

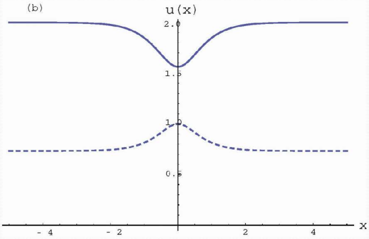
$$(\rho u)_x = 0,$$
$$u u_x + \rho_x + V_x(x) = 0,$$

Solution

$$\rho u = v, \quad \tfrac{1}{2}u^2 + \rho + V(x) = \tfrac{1}{2}v^2 + 1\,,$$

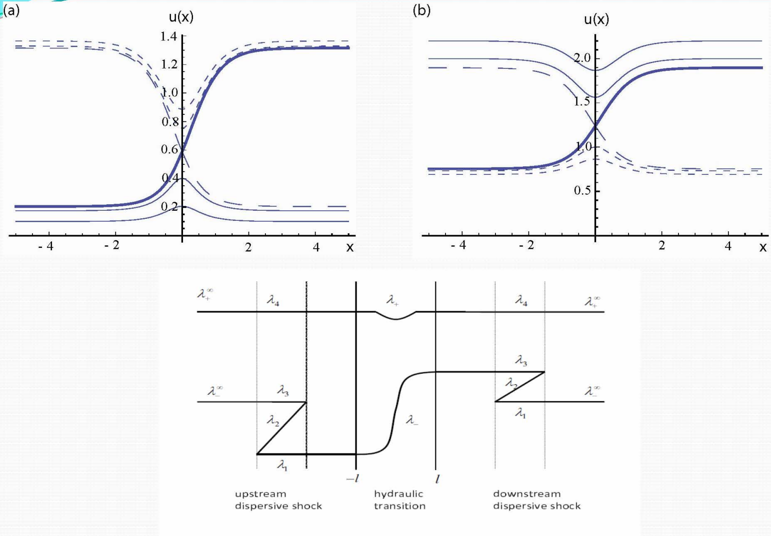


$v < v_-$

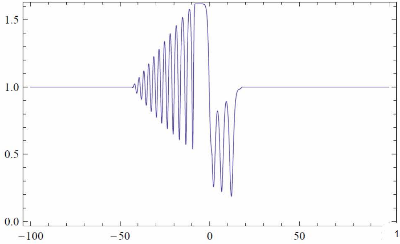


$v > v_+$

Two dispersive shocks: upstream and downstream

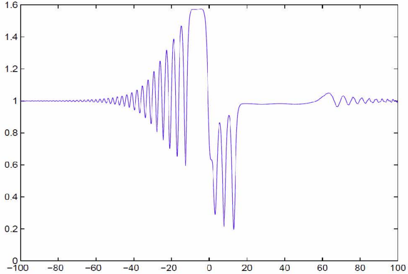


Comparison of analytical theory with numerical solution



← Analytical theory

Numerical solution →



Polariton condensate

Polariton dynamics is described by the Gross-Pitaevskii equation with account of pumping and dissipation

$$i\psi_t + \frac{1}{2}\psi_{xx} - |\psi|^2\psi = V(x)\psi + i(\gamma - \Gamma|\psi|^2)\psi,$$

Periodic solution ($\rho = |\psi|^2$)

$$\rho = \frac{1}{4}(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)^2 + (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3) \\ \times \mathrm{sn}^2(\sqrt{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)}\,\theta, m),$$

where

$$\theta = x - Ut, \quad U = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$$

Whitham equations

$$\frac{d\lambda_i}{dx} = \frac{2}{L} \cdot \frac{I_1\lambda_i + I_2}{\prod_{m\neq i}(\lambda_i - \lambda_m)},$$

where

$$I_1 = \Gamma \int_{\nu_1}^{\nu_2} \frac{\nu(\rho_0 - \nu)}{\sqrt{\mathcal{R}(\nu)}} d\nu, \quad I_2 = \frac{\Gamma u_0 \rho_0}{2} \int_{\nu_1}^{\nu_2} \frac{\rho_0 - \nu}{\sqrt{\mathcal{R}(\nu)}} d\nu,$$

and

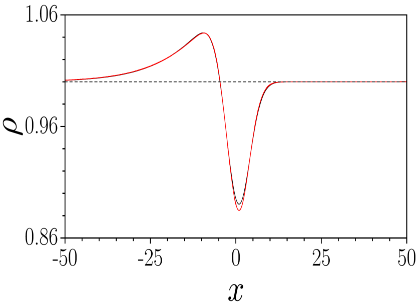
$$\mathcal{R} = (\nu - \nu_1)(\nu - \nu_2)(\nu - \nu_3)$$
$$\nu_1 = \tfrac{1}{4}(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)^2,$$
$$\nu_2 = \tfrac{1}{4}(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)^2,$$
$$\nu_3 = \tfrac{1}{4}(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)^2,$$

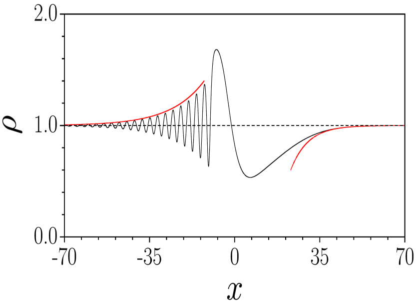
The system has two integrals:

$$\frac{ds_1}{dx} = 0, \quad \frac{ds_2}{dx} = 0, \quad \frac{ds_3}{dx} = \frac{2I_1}{L}, \quad \frac{ds_4}{dx} = -\frac{2I_2}{L}.$$

Solution for the tail of the envelope

$$\delta\nu_1, \delta\nu_2 \propto \exp\left(\frac{\Gamma\rho_0 u_0}{u_0^2 - \rho_0} x\right),$$





Thank you for your attention!