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JUSTIFICATION OF ADIABATIC PRINCIPLE FOR HYPERBOLIC GINZBURG–LANDAU EQUATIONS

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Hyperbolic Ginzburg–Landau equations are Euler–Lagrange equations for

$(2+1)$-dimensional Abelian Higgs model

This model is governed by the *Ginzburg–Landau action*

$$S(\mathcal{A}, \Phi) = \int_0^{T_0} \left(T(\mathcal{A}, \Phi) - U(\mathcal{A}, \Phi)\right) dt$$

on the space $\mathbb{R}^{1+2}$ with coordinates $(x_0 = t, x_1, x_2)$.

The action $S(\mathcal{A}, \Phi)$ depends on variables $\mathcal{A}$ and Φ where

$\mathcal{A} = A_0\, dt + A_1 dx_1 + A_2\, dx_2 =: A^0 + A$ is a 1-form on $\mathbb{R}^{1+2}$ with smooth purely imaginary coefficients

$$A_\mu = A_\mu(t, x_1, x_2), \qquad \mu = 0, 1, 2$$

Φ is the *Higgs field*, given by a smooth complex-valued function

$$\Phi = \Phi(t, x_1, x_2) = \Phi_1 + i\Phi_2 \qquad \text{on } \mathbb{R}^{1+2}.$$

Physically,

- $\mathcal{A}$ is vector-potential of electromagnetic field
- Φ is a scalar field, interacting with this electromagnetic field.

Potential energy:

$$U(\mathcal{A},\Phi)=\frac{1}{2}\int\left\{|F_A|^2+|d_A\Phi|^2+\frac{1}{4}(1-|\Phi|^2)^2\right\}dx_1\,dx_2,$$

where

F_A is the 2-form $\quad F_A = dA = \sum_{i,j=1}^{2} F_{ij}\,dx_i\wedge dx_j$

with coefficients $\quad F_{ij}=\partial_i A_j-\partial_j A_i, \qquad \partial_j := \partial/\partial x_j\,, \quad i,j=1,2$

(where only two of them, i.e. F_{12} and $F_{21}=-F_{12}$, do not vanish).

covariant derivative $\quad d_A\Phi = d\Phi + A\,\Phi = \sum_{i=1}^{2}(\partial_i + A_i)\Phi\,dx_i.$

Physically,

- F_A represents electromagnetic field and $|F_A|^2$ is the Maxwell tensor;
- $|d_A\Phi|^2$ is responsible for the interaction of electromagnetic field with Higgs field Φ;
- $\frac{1}{4}(1-|\Phi|^2)^2$ reflects a non-linear character of Φ.

Note that potential energy depends only on space components of $\mathcal{A}$ so

$$U(\mathcal{A},\Phi)=U(A,\Phi).$$

Condition $U(A, \Phi) < \infty$ $\Longrightarrow$ the existence of limit $|\Phi| \to 1$ at space infinity

$\Downarrow$

the map $\Phi \colon \mathbb{R}^2 \longrightarrow \mathbb{C}$ sends circles S^1_R of sufficiently large radius R to topological circles $\Longrightarrow$ Φ has a topological index called *vortex number.*

Vortex number is equal to the algebraic sum of indices of zeros of Φ inside S^1_R.

Kinetic energy:

$$T(\mathcal{A}, \Phi) = \frac{1}{2} \int \left\{ 2|F_{01}|^2 + 2|F_{02}|^2 + |d_{A^0}\Phi|^2 \right\} dx_1 \, dx_2$$

where

- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu , \qquad \partial_\mu := \partial / \partial x_\mu , \quad \mu, \nu = 0, 1, 2$
- $d_{A^0}\Phi = d\Phi + A_0 \, dt.$

Ginzburg–Landau equations are Euler–Lagrange equations for the action $S(\mathcal{A}, \Phi)$:

$$\begin{cases} \partial_0 F_{0j} + \sum_{k=1}^{2} \varepsilon_{jk} \partial_k F_{12} = i\,\mathrm{Im}(\bar{\Phi} \nabla_{A,j} \Phi), \qquad j = 1, 2 \\ \partial_1 F_{01} + \partial_2 F_{02} = i\,\mathrm{Im}(\bar{\Phi} \nabla_{A,0} \Phi) \\ (\nabla_{A,0}^2 - \nabla_{A,1}^2 - \nabla_{A,2}^2)\Phi = \frac{1}{2}\Phi(1 - |\Phi|^2), \end{cases}$$

where $\qquad \nabla_{A,\mu} = \partial_\mu + A_\mu, \quad \mu = 0, 1, 2; \qquad \varepsilon_{12} = -\varepsilon_{21} = 1, \quad \varepsilon_{11} = \varepsilon_{22} = 0.$

Ginzburg–Landau equations, as well as the action $S(\mathcal{A}, \Phi)$, are invariant under *gauge transforms*

$$\begin{aligned} A_\mu &\longmapsto A_\mu + i\partial_\mu \chi, \\ \Phi &\longmapsto e^{-i\chi}\Phi, \end{aligned} \qquad \mu = 0, 1, 2\,,$$

where χ is a smooth real-valued function on $\mathbb{R}^{1+2}$.

Solutions of Ginzburg–Landau equations are called *dynamic solutions.*

We are interested in description of the *moduli space of dynamic solutions,* equal to quotient of the space of solutions of Ginzburg–Landau equations modulo gauge transforms.

Static solutions – vortices

Static solutions of Ginzburg–Landau equations realize local minima of potential energy $U(A,\Phi)$.

Static solutions with vortex number $N>0$ are called *N-vortices* and characterized by the following theorem of Taubes.

Introduce complex coordinate $z = x_1 + ix_2$ on the plane $\mathbb{R}^2_{(x_1,x_2)}$.

Theorem (Taubes)**.** *For any unordered collection $z_1,\dots,z_N$ of N points on $\mathbb{C}$, some of which may coincide, there exists a unique (up to static gauge transforms) N-vortex solution (A,Φ) such that Φ vanishes precisely at points $z_1,\dots,z_N$ with prescribed multiplicities.*

This Theorem implies that the *moduli space of N-vortices,* defined as

$$\mathcal{M}_N = \frac{\{N\text{-vortices } (A,\Phi)\}}{\{\text{static gauge transforms}\}}$$

may be identified with Nth symmetric power of the complex plane $\mathbb{C}$:

$$\mathcal{M}_d = \mathrm{Sym}^N \mathbb{C},$$

i.e. the space of unordered collections of N points on $\mathbb{C}$.

$\mathrm{Sym}^N\mathbb{C}$ may be identified with $\mathbb{C}^N$ by associating with collection $(z_1,\dots,z_N)$ the monic polynomial, having zeros at these points with given multiplicities.

Adiabatic limit

For an appropriate choice of gauge function χ we can always achieve the condition

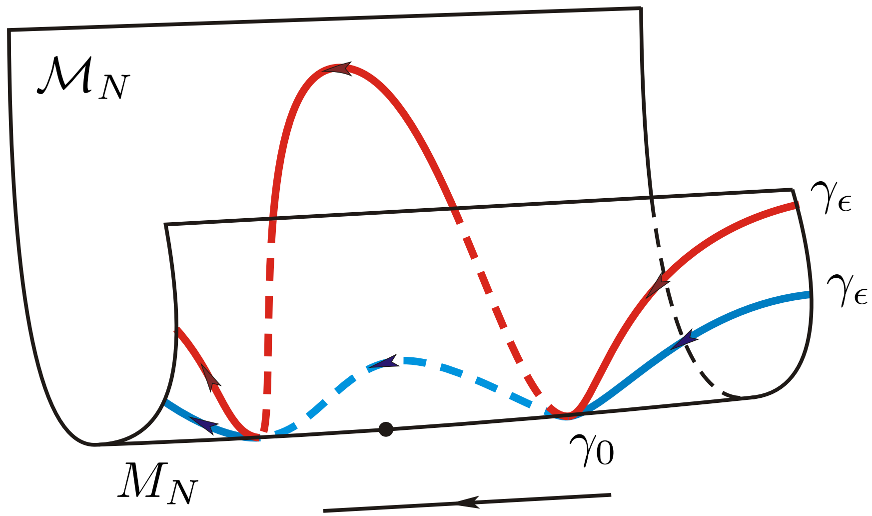
$$\boxed{A_0 = 0.}$$

Such a choice of χ is called the *temporal gauge.* (Note that, after imposing this condition on χ, we still have gauge freedom with respect to static gauge transforms, given by gauge functions χ, depending only on space variables x_1, x_2).

In temporal gauge a dynamic solution of Ginzburg–Landau equations can be considered as a trajectory of the form

$$\gamma : t \longmapsto [A(t), \Phi(t)]$$

where $[A, \Phi]$ denotes the gauge class of (A, Φ) with respect to static gauge transforms.



This trajectory lies in *configuration space*

$$\mathcal{N}_N = \frac{\{(A, \Phi) \text{ with } U(A, \Phi) < \infty \text{ and vortex number } N\}}{\{\text{static gauge transforms}\}},$$

containing, in particular, the moduli space of N-vortices $\mathcal{M}_N$.

Consider a sequence of dynamic solutions γ_ϵ, depending on $\epsilon > 0$, given by trajectories

$$\gamma_\epsilon\colon t \longmapsto [A_\epsilon(t), \Phi_\epsilon(t)].$$

Suppose that their kinetic energy

$$T(\gamma_\epsilon) := \int_0^{T_0} T(\gamma_\epsilon(t))dt \approx \epsilon$$

tends to zero proportionally to ϵ when $\epsilon \to 0$.

Then in the limit $\epsilon \to 0$ trajectory γ_ϵ converts into a static solution, i.e. a point of $\mathcal{M}_N$. However, if we introduce a "slow time" parameter $\tau = \epsilon t$ on γ_ϵ and consider the limit of "rescaled" trajectories $\gamma_\epsilon(\tau)$ for $\epsilon \to 0$ then in the limit we obtain a trajectory γ_0, lying in $\mathcal{M}_N$, rather than a point.

Such a limit is called *adiabatic* and γ_0 is called *adiabatic trajectory*:

$$\gamma_0 = \text{adiabatic limit of } \gamma_\epsilon \text{ for } \epsilon \to 0.$$

Adiabatic trajectories admit the following intrinsic description in terms of $\mathcal{M}_N$.

Theorem. *Kinetic energy functional determines a Riemannian metric (T-metric) on $\mathcal{M}_N$ with geodesics given by the adiabatic trajectories γ_0.*

Note that every point of an adiabatic trajectory γ_0 is a static solution so γ_0 as a whole cannot be a dynamic solution. However, one can think of this trajectory as describing approximately some dynamic solution with small kinetic energy.

The idea of approximate decription of "slow" dynamic solutions in terms of the moduli space of static solutions is due to Manton. Since we know very little about the structure of moduli space of dynamic solutions, this approach looks rather perspective.

Manton has also proposed the following heuristic *adiabatic principle:*

for any adiabatic trajectory γ_0 on the moduli space of N-vortices $\mathcal{M}_N$ it should exist a sequence $\{\gamma_\epsilon\}$ of dynamic solutions, tending to γ_0 in adiabatic limit.

The main goal of my talk is to explain mathematical meaning of this principle.

Tangent structure of $\mathcal{M}_N$

We study first in more detail the tangent structure of moduli space of N-vortices $\mathcal{M}_N$.

It was shown by Taubes that vortices are solutions (with finite energy) of the following *vortex equations*

$$\begin{cases} \bar\partial_A \Phi = 0 \\ iF_A = *\frac{1}{2}\left(1 - |\Phi|^2\right) \end{cases}$$

where

$*$ is the Hodge operator on $\mathbb{R}^2_{(x_1,x_2)}$,

$\bar\partial_A := \bar\partial_z + A^{0,1}$ where A is written in complex coordinate: $A = A^{1,0} + A^{0,1}$.

Second vortex equation may be also rewritten as: $iF_{12} = \frac{1}{2}\left(1 - |\Phi|^2\right)$.

Tangent space to $\mathcal{M}_N$ at a point, corresponding to N-vortex (A, Φ), is spanned by solutions of *linearized vortex equations:*

$$\begin{cases} \bar\partial_A \varphi + a^{0,1}\Phi = 0 \\ *i\,da + \mathrm{Re}(\varphi\bar\Phi) = 0 \end{cases}$$

where

a is a 1-form with smooth purely imaginary coefficients,

φ is a smooth complex-valued function on $\mathbb{R}^2$.

Solutions of linearized vortex equations are considered up to *infinitesimal gauge transforms:*

$$(a, \varphi) \longmapsto (a + i\, d\chi,\ \varphi - i\Phi\chi)$$

where χ is a smooth real-valued function on $\mathbb{R}^2$.

The left-hand sides of linearized vortex equations determine the *linearized vortex operator*

$$D_{(A,\Phi)}(a, \varphi) = \big(\bar{\partial}_A \varphi + a^{0,1}\Phi,\ *i\, da + \mathrm{Re}(\varphi\bar{\Phi})\big).$$

In its terms the tangent space to $\mathcal{M}_N$ at a given point (A, Φ) is written in the form

$$T_{(A,\Phi)}\mathcal{M}_N = \frac{\{(a, \varphi) \colon D_{(A,\Phi)}(a, \varphi) = 0\}}{\{\text{infinitesimal gauge transforms}\}}\ .$$

One can get rid of factorization modulo infinitesimal gauge transforms by gauge fixing. We use for that the operator δ_Φ^*, adjoint to *gauge operator*

$$\delta_\Phi \colon \chi \longmapsto (a + i\, d\chi,\ \varphi - i\Phi\chi).$$

In other words, we fix the infitesimal gauge by imposing the following condition

$$\delta_\Phi^*(a, \varphi) = 0.$$

Main theorem

Theorem (Palvelev). *Suppose that a trajectory*

$$\gamma_0 = [A_0, \Phi_0] \colon [0, \tau_0] \longrightarrow \mathcal{M}_N$$

is a geodesic on $\mathcal{M}_N$ with respect to kinetic T-metric, provided with the natural parameter τ. Then there exists its pull-back

$$(A_0, \Phi_0) \colon [0, \tau_0] \longrightarrow \{N\text{-vortices}\}$$

to the space of N-vortices, given by a smooth trajectory (A_0, Φ_0), and positive constants $\tau_1 \leqslant \tau_0$, ϵ_0, M such that for any $\epsilon < \epsilon_0$ there exists a dynamic solution $(\mathcal{A}^\epsilon, \Phi^\epsilon)$ of Ginzburg–Landau equations on interval $[0, \tau_1/\epsilon]$, having the form

$$\begin{cases} A_0^\epsilon = \epsilon^3 a_0, \\ A^\epsilon(t) = A_0(\epsilon t) + \epsilon^2 a(t) \equiv A(\epsilon t) + \epsilon^2 a(t), \\ \Phi^\epsilon(t) = \Phi_0(\epsilon t) + \epsilon^2 \varphi(t) \equiv \Phi(\epsilon t) + \epsilon^2 \varphi(t), \end{cases}$$

and satisfying the estimate

$$\max \{\|a_0(t)\|_{H^3}, \|a(t)\|_{H^3}, \|\varphi(t)\|_{H^3}\} \leqslant M$$

for any $t \in [0, \tau_1/\epsilon]$.

The norm $\|\cdot\|_{H^3}$ denotes the Sobolev H^3-norm on $\mathbb{R}^2$.

Auxiliary system

Plugging the above Ansatz into Ginzburg–Landau equations, we obtain the following system of equations

$$\begin{cases} -\Delta a_0 + |\Phi|^2 a_0 = \dfrac{1}{\epsilon^2}\,\delta_\Phi^*(\partial_\tau\psi) + \dfrac{1}{\epsilon}\,\partial_t(\delta_\Phi^*\psi) + 2(\partial_\tau\Phi_2\cdot\varphi_1 - \partial_\tau\Phi_1\cdot\varphi_2) + \cdots\,, \\ \partial_t^2\psi + D^*_{(A,\Phi)}D_{(A,\Phi)}\psi = -\partial_\tau^2\psi + \cdots \end{cases}$$

where $\psi = (a, \varphi)$ and "$\cdots$" denote the terms, proportional to ϵ.

Due to gauge invariance, operator $D_{(A,\Phi)}$ has infinite-dimensional kernel. In order to avoid infinite-dimensional degeneration, we replace this operator by a perturbed operator

$$\mathcal{D}_{(A,\Phi)} = D_{(A,\Phi)} \oplus \delta_\Phi^*,$$

containing the adjoint gauge operator.

To get rid of terms, proportional to $1/\epsilon^2$ and $1/\epsilon$ in the right-hand side of the first equation, we fix the gauge by imposing the condition $\delta_\Phi^*\psi = 0$.

So we replace the original system by the following auxiliary system

$$\begin{cases} -\Delta a_0 + |\Phi|^2 a_0 = 2(\partial_\tau \Phi_2 \cdot \varphi_1 - \partial_\tau \Phi_1 \cdot \varphi_2) + \cdots \,, \\ \partial_t^2 \psi + \mathcal{D}^*_{(A,\Phi)} \mathcal{D}_{(A,\Phi)} \psi = -\partial_\tau^2 \psi + \cdots \,, \end{cases}$$

which has already only finite-dimensional degeneration.

This replacement of the original system by the auxiliary one is justified by the following

Theorem. *Any solution of the auxiliary system, satisfying initial gauge fixing condition*

$$\delta^*_\Phi \psi = 0, \quad \partial_t \delta^*_\Phi \psi = 0 \quad \text{for } t = t_0,$$

satisfies also the original system.

Thus, the existence problem for the original system is reduced to the existence problem for the covariant Cauchy problem for auxiliary system.

Solving the auxiliary system

To solve the existence problem for auxiliary system, we shall make use of the following two results where we adopt a notation: $\vec{a}_0 := (a_0, \partial_t a_0)$, $\quad \vec{\psi} := (\psi, \partial_t \psi)$.

Theorem (local existence theorem). *For any sufficiently small $\epsilon > 0$ and arbitrary initial conditions*

$$\vec{\psi}(t_0) = \vec{\psi}^{\,0}$$

there exists an interval $[t_0, t_0 + \delta_0]$ and solution of auxiliary system on this interval, satisfying the estimate:

$$\max\left\{\|\vec{a}_0(t)\|, \|\vec{\psi}(t)\|\right\} \leqslant B_1\|\vec{\psi}^{\,0}\| + B_2 \quad \textit{for any } t \in [t_0, t_0 + \delta_0].$$

A key role in the proof of main Theorem is played by the following

Theorem (long-time apriori estimate). *Suppose that (a_0, ψ) is a solution of our system on interval $[0, T_0]$ with zero initial condition: $\vec{\psi}(0) = 0$. Assume that this solution satisfies the estimate*

$$\max\left\{\|\vec{a}_0(t)\|, \|\vec{\psi}(t)\|\right\} \leqslant M \quad \textit{for any } t \in [0, T_0].$$

Then for any $\epsilon < 1/M$ and all $t \in [0, T_0]$ the following estimate is true

$$\|\vec{\psi}(t)\| \leqslant C_1 + \epsilon t C_2$$

where $C_1 = \text{const} > 0$, $C_2 = C_1(1 + M^4)$.

Strategy of constructing a solution of auxiliary system

Existence of a solution is proved by successive usage of local existence theorem and long-time apriori estimate.

First we choose constants M and τ_1 from main Theorem so that M satisfies the inequality

$$M > B_1 C_1 + B_2,$$

and τ_1 satisfies the equation

$$B_1 C_1 \big(1 + \tau_1 (1 + M^4)\big) + B_2 = M.$$

We have to show that for any sufficiently small $\epsilon > 0$ there exists a solution of our system on interval $[0, \tau_1/\epsilon]$.

Step 1. Using local existence theorem, we find first a solution of our system with zero initial condition on a small interval $[0, \delta_0]$. This solution satisfies the estimate

$$\max\{\|\vec{a}_0(t)\|, \|\vec{\psi}(t)\|\} < M \quad \text{for any } t \in [0, \delta_0]$$

(since $M > B_2$).

If $\delta_0 > \tau_1/\epsilon$ then the process is over. If not then, according to apriori estimate, we have the inequality

$$\|\vec{\psi}(t)\| \leqslant C_1 + \epsilon t C_2 \quad \text{for any } t \in [0, \delta_0].$$

Step 2. Using local existence theorem, we find next a solution of our system on interval $[\delta_0, 2\delta_0]$ with initial condition $\vec{\psi}^{\,0} = \vec{\psi}(\delta_0)$.

For this initial condition, according to Step 1, we have the estimate

$$\|\vec{\psi}^{\,0}\| \leqslant C_1 + \epsilon\delta_0 C_2 \leqslant C_1 + \tau_1 C_2.$$

Hence, by local existence theorem, this solution satisfies the estimate

$$\max\{\|\vec{a}_0(t)\|, \|\vec{\psi}(t)\|\} \leqslant$$
$$\leqslant B_1\|\vec{\psi}(\delta_0)\| + B_2$$
$$\leqslant B_1 C_1 + \tau_1 B_1 C_2 + B_2 = M$$

(in last inequality we have used the definition of τ_1).

Step 3. Consider now the combined solution of our system on interval $[0, 2\delta_0]$ with zero initial condition, composed of solutions on intervals $[0, \delta_0]$ and $[\delta_0, 2\delta_0]$, constructed before. (Of course, one have to check that this combined solution does satisfy our system on the whole interval $[0, 2\delta_0]$.)

According to Steps 1 and 2, this solution satisfies the estimate

$$\max\{\|\vec{a}_0(t)\|, \|\vec{\psi}(t)\|\} \leqslant M \quad \text{for any } t \in [0, 2\delta_0].$$

If $2\delta_0 > \tau_1/\epsilon$ then the process is over. If not then, according to apriori estimate, we have:

$$\|\vec{\psi}(t)\| \leqslant C_1 + \epsilon t C_2 \quad \text{for any } t \in [0, 2\delta_0].$$

Step 4. We find now a solution of our system on the next interval $[2\delta_0, 3\delta_0]$ with initial condition $\vec{\psi}^{\,0} = \vec{\psi}(2\delta_0)$ and apply to this solution the argument from Steps 2 and 3.

This process can be repeated until at some kth step we get the inequality $k\delta_0 > \tau_1/\epsilon$. At this step the process is over, as well as the proof of main Theorem.