Self-adjoint commuting differential operators and commutative subalgebras of the Weyl algebra A_1

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Weyl algebra A_1 over field k of characteristic zero is given by two generators p, q and one relation

$$[p, q] = 1.$$

Weyl algebra over $\mathbb{C}$ can be considered as a ring of differential operators with polynomial coefficients:

$$q = x,\ p = \partial_x.$$

Theorem (J. Dixmier, 1968) *In* A_1 *there is a maximal commutative subalgebra: elements*

$$X = (p^3 + q^2 + h)^2 + 2p,$$

$$Y = (p^3 + q^2 + h)^3 + \frac{3}{2}\Big(p(p^3 + q^2 + h) + (p^3 + q^2 + h)p\Big)$$

commute, $h \in k$*, herewith*

$$Y^3 = X^2 - h.$$

$$L_1 = \frac{d^n}{dx^n} + u_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + u_0(x),$$

$$L_2 = \frac{d^m}{dx^m} + v_{m-1}(x)\frac{d^{m-1}}{dx^{m-1}} + \cdots + v_0(x).$$

Lemma (Schur, 1905)

If $L_1L_2 = L_2L_1$ *and* $L_1L_3 = L_3L_1$ *(*L_1 *is non-trivial), then*

$$L_2L_3 = L_3L_2.$$

Lemma (Burchnall, Chaundy, 1923)

If $L_1L_2 = L_2L_1$*, then there exist a non-trivial polynomial* $Q(\lambda, \mu)$ *of two commuting variables such that* $Q(L_1, L_2) = 0$*.*

Example

$$L_1 = \frac{d^2}{dx^2} - \frac{2}{x^2}, \quad L_2 = \frac{d^3}{dx^3} - \frac{3}{x^2}\frac{d}{dx} + \frac{3}{x^3}$$

$$L_1^3 = L_2^2, \quad Q(\lambda,\mu) = \lambda^3 - \mu^2.$$

Spectral curve

$$\Gamma = \{(\lambda,\mu) \in \mathbb{C}^2 : Q(\lambda,\mu) = 0\}.$$

If $L_1\psi = \lambda\psi$ and $L_2\psi = \mu\psi$, then $(\lambda,\mu) \in \Gamma$.

rank of L_1 and L_2 is

$$l = \dim\{\psi : L_1\psi = \lambda\psi,\ L_2\psi = \mu\psi\}.$$

Baker–Akhiezer function $\psi(x, P)$

Spectral data

$$\{\Gamma, q, k^{-1}, \gamma_1, \ldots, \gamma_g\}$$

Γ is algebraic curve, $q \in \Gamma$, k^{-1} is a local parameter near q, $k^{-1}(q) = 0$, $\gamma_1, \ldots \gamma_g \in \Gamma$.

The Baker–Akhiezer function has the property:

1. $\psi = e^{kx}\left(1 + \frac{f(x)}{k} + \ldots\right)$

2. on $\Gamma \backslash q$ the BA-function ψ is meromorphic with the poles in $\gamma_1, \ldots, \gamma_g$

For $\gamma_1, \ldots, \gamma_g$ in general position the BA-function ψ there exists and unique.

Let f(P) be a meromorphic function on Γ with a unique pole in q of order n

$$f = k^n + c_{n-1}k^{n-1} + \cdots + c_0 + \frac{c_{-1}}{k} + \ldots$$

$$\partial_x^n \psi = k^n e^{kx}(O(1)),$$

$$\partial_x^n \psi - f\psi = k^{n-1}e^{kx}\left(u_{n-1}(x) + O\left(\frac{1}{k}\right)\right),$$

$$\partial_x^n \psi + u_{n-1}(x)\partial_x^{n-1}\psi - f\psi = k^{n-2}e^{kx}\left(u_{n-2}(x) + O\left(\frac{1}{k}\right)\right),$$

$$\partial_x^n \psi + u_{n-1}(x)\partial_x^{n-1}\psi + \cdots + u_0(x)\psi = f\psi + e^{kx}\left(O\left(\frac{1}{k}\right)\right).$$

From the uniqueness of BA-function it follows that

$$L_1\psi(x,P) = f(p)\psi(x,P).$$

Let g(P) be a meromorphic function on Γ with unique pole in q of order m, then

$$L_2\psi(x,P) = g(P)\psi(x,P).$$

We have

$$(L_1L_2 - L_2L_1)\psi(x,P) = 0 \Rightarrow L_1L_2 = L_2L_1.$$

Example $\Gamma = \mathbb{C}P^1,\ q = \infty,\ k = z$

Baker–Akhiezer function $\psi = e^{xz}$

$$f = z^n + c_{n-1}z^{n-1} + \cdots + c_0,$$

$$\partial_x^n\psi + c_{n-1}\partial_x^{n-1}\psi + \cdots + c_0\psi = f\psi.$$

Example

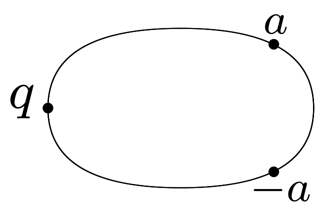
$$\Gamma = \mathbb{C}/\{2\omega\mathbb{Z} + 2\omega'\mathbb{Z}\}, \;\; q = 0,$$

$$\psi = e^{-x\zeta(z)}\frac{\sigma(z+x)}{\sigma(x)\sigma(z)},$$

$$(\partial_x^2 - 2\wp(x))\psi(x,z) = \wp(z)\psi(x,z),$$

$$\left(\partial_x^3 - 3\wp(x)\partial_x - \frac{3}{2}\wp'(x)\right)\psi(x,z) = \frac{1}{2}\wp'(z)\psi(x,z).$$

Example $\Gamma = \mathbb{C}P^1/\{a \sim -a\},\ q = \infty,\ g_a = 1,\ k = z$



$$\psi = e^{xz}\left(1 + \frac{\xi(x)}{z-\gamma}\right),$$

$$\psi(a) = \psi(-a) \Rightarrow \xi(x) = \frac{(\gamma^2 - a^2)\sinh(ax)}{a\cosh(ax) + \gamma\sinh(ax)},$$

$$(\partial_x^2 - u(x))\psi = z^2\psi,\ u(x) = -\frac{2a^2(a^2-\gamma^2)}{(a\cosh(ax) + \gamma\sinh(ax))^2}.$$

Rank $l > 1$

Spectral data

$$\{\Gamma, q, k^{-1}, \gamma_1, \ldots, \gamma_{lg}, \alpha_1, \ldots, \alpha_{lg}\}$$

$\alpha_i = (\alpha_{1i}, \ldots, \alpha_{il-1})$ — vector

(γ, α) — Turin parameters define stable (in the sense of Mumford) vector bundle of rank l degree lg on Γ with holomorphic sections $\eta_1, \ldots, \eta_l$

$$\eta_l(\gamma_i) = \sum_{j=1}^{l-1} \alpha_{ij} \eta_j(\gamma_i).$$

Vector Baker–Akhiezer function $\psi(x,P)=(\psi_0(x,P),\dots,\psi_{l-1}(x,P))$:

1. $\psi(x,P)=\left(\sum_{s=0}^{\infty}\xi_s(x)k^{-s}\right)\Psi_0(x,P)$, $\xi_0=(1,0,\dots,0)$, $\frac{d}{dx}\Psi_0=A\Psi_0$,

$$A=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ k+u_0(x) & u_1(x) & u_2(x) & \dots & u_{l-1}(x) & 0 \end{pmatrix}$$

2. on $\Gamma-\{q\}$ ψ is meromorphic with the simple poles in $\gamma_1,\dots,\gamma_{lg}$

3. $\mathrm{Res}_{\gamma_i}\psi_j=\alpha_{ij}\mathrm{Res}_{\gamma_i}\psi_{l-1}$.

If f(P) is meromorphic function with the pole in q of order n, then there exist $L(f)$ such that

$$L(f)\psi(x,P)=f(P)\psi(x,P),\ \mathrm{ord}L(f)=ln.$$

Method of Turin parameters deformation (Krichever–Novikov method)

$$\frac{d^l}{dx^l}\psi_j = \chi_{l-1}\frac{d^{l-1}}{dx^{l-1}}\psi_j + \cdots + \chi_0\psi_j$$

χ_s — meromorphic on Γ, χ_s has lg simple poles $P_1(x),\dots,P_{lg}(x)$. In the neighbourhood of q the functions χ_s have the form

$$\chi_0(x,P) = k + g_0(x) + O(k^{-1}),$$

$$\chi_j(x,P) = g_j(x) + O(k^{-1}), \quad j < l-1,$$

$$\chi_{l-1}(x,P) = O(k^{-1}).$$

At the point $P_i(x)$

$$\chi_j = \frac{c_{ij}(x)}{k - \gamma_i(x)} + d_{ij}(x) + O(k - \gamma_i(x)).$$

Theorem *Parameters* $\gamma_i(x), \alpha_{ij}(x) = \frac{c_{ij}(x)}{c_{i,l-1}(x)}$, *and* $d_{ij}(x), 0 \le j \le l - 2, 1 \le i \le lg$ *satisfy the equation*

$$c_{i,l-1}(x) = -\gamma_i'(x),$$

$$d_{i0}(x) = \alpha_{i0}(x)\alpha_{i,l-2}(x) + \alpha_{i0}(x)d_{i,l-1}(x) - \alpha_{i0}'(x),$$

$$d_{ij}(x) = \alpha_{ij}(x)\alpha_{i,l-2}(x) - \alpha_{i,j-1}(x) + \alpha_{ij}(x)d_{i,l-1}(x) - \alpha_{ij}'(x), j \ge 1.$$

Krichever, Novikov: $g=1,\ l=2$ $\Gamma: \mu^2 = P_3(\lambda) = 4\lambda^3 + g_2\lambda + g_3$

$$L_1 = \left(\partial_x^2 + u\right)^2 + 2c_x(\wp(\gamma_2) - \wp(\gamma_1))\partial_x + (c_x(\wp(\gamma_2) - \wp(\gamma_1)))_x - \wp(\gamma_2) - \wp(\gamma_1),$$

$$\gamma_1(x) = \gamma_0 + c(x),\ \gamma_2(x) = \gamma_0 - c(x),$$

$$u(x) = -\frac{1}{4c_x^2} + \frac{1}{2}\frac{c_{xx}^2}{c_x^2} + 2\Phi(\gamma_1,\gamma_2)c_x - \frac{c_{xxx}}{2c_x} + c_x^2(\Phi_c(\gamma_0+c,\gamma_0-c) - \Phi^2(\gamma_1,\gamma_2)),$$

$$\Phi(\gamma_1,\gamma_2) = \zeta(\gamma_2-\gamma_1) + \zeta(\gamma_1) - \zeta(\gamma_2).$$

Operator L_2 can be find from the equation $L_2^2 = P_3(L_1)$.

Dixmier: $g=1,\ l=2$

$$L_1 = \left(\frac{d^2}{dx^2} - x^3 - \alpha\right)^2 - 2x,$$

$$L_2 = \left(\frac{d^2}{dx^2} - x^3 - \alpha\right)^3 - \frac{3}{2}\left(x\left(\frac{d^2}{dx^2} - x^3 - \alpha\right) + \left(\frac{d^2}{dx^2} - x^3 - \alpha\right)x\right).$$

Theorem (Grinevich)

Commuting operators L_1 and L_2 corresponding to an elliptic curve have rational coefficients if and only if

$$c(x) = \int_{q(x)}^{\infty} \frac{dt}{\sqrt{P_3(t)}},$$

where $q(t)$ is a rational function.

If $\gamma_0 = 0$, and $q(x) = x$, we have the Dixmier operators.

Theorem (Grinevich, Novikov) *Operator L_1 is formally self-adjoint if and only if $\wp(\gamma_1) = \wp(\gamma_2)$.*

Mokhov: $g = 1, \; l = 3$

Rank $l = 2,\ g > 1$: self-adjoint case

Let L be an operator of the forth order of rank 2, then

$$\Gamma : w^2 = F(z) = z^{2g+1} + c_{2g}z^{2g} + \cdots + c_0,$$

q is a branch point,

$$\sigma : \Gamma \to \Gamma,\ \sigma(z, w) = (z, -w).$$

We have

$$\psi'' = \chi_0\psi + \chi_1\psi',$$

where $\psi = (\psi_1, \psi_2)$ is a Baker–Akhiezer function.

Theorem (M.) *If*

$$\chi_1(x, P) = \chi_1(x, \sigma(P)),$$

then operator L is self-adjoint

$$L = L^* = (\partial_x^2 + V(x))^2 + W(x).$$

If $g = 2, 3, 4$, and L is self-adjoint then $\chi_1(x, P) = \chi_1(x, \sigma(P))$.

$$w^2 = F(z) = z^{2g+1} + c_{2g} z^{2g} + \cdots + c_0,$$

$$\chi_0 = -\frac{1}{2}\frac{H_1(x)\gamma_1'(x)}{z-\gamma_1(x)} - \cdots - \frac{1}{2}\frac{H_g(x)\gamma_g'(x)}{z-\gamma_g(x)} + \frac{w}{2(z-\gamma_1)\dots(z-\gamma_g)} + \frac{\kappa(x)}{2},$$

$$\chi_1(x,P) = -\frac{\gamma_1'(x)}{z-\gamma_1(x)} - \cdots - \frac{\gamma_g'(x)}{z-\gamma_g(x)}.$$

We have $2g$ equations on $\gamma_1, \ldots, \gamma_g, H_1, \ldots, H_g, \kappa$

$$L_i = d_{i0}(x) - (\alpha_i^2(x) + \alpha_i(x)d_{i1}(x) - \alpha_i'(x)) = 0, \ 1 \le i \le 2g.$$

These equations can be reduced the $g-1$ equations on $\gamma_1, \ldots, \gamma_g$. From

$$L_i - L_{i+g} = 0$$

one can express H_i in terms of $\gamma_1, \ldots \gamma_g$. From

$$L_i + L_{i+g} = 0$$

one can express κ in terms of $\gamma_1, \ldots \gamma_g$, and H_i. We have $g-1$ equations on $\gamma_1, \ldots, \gamma_g$. Operator L has the form

$$L = L^* = (\partial_x^2 + V(x))^2 + W(x),$$

where

$$V = -\frac{\kappa}{2}, \ W = -\frac{1}{2}(\gamma_1 + \cdots + \gamma_g).$$

Example: $g = 1$

$$w^2 = F(z) = z^3 + c_2 z^2 + c_1 z + c_0.$$

$$\chi_0 = -\frac{H_1(x)\gamma_1'(x)}{z - \gamma_1(x)} + \frac{w(z)}{z - \gamma_1(x)} + \kappa(x), \;\; \chi_1(x, P) = -\frac{\gamma_1'(x)}{z - \gamma_1(x)},$$

$$H_1(x) = -\frac{\gamma_1''(x)}{2\gamma_1'(x)}, \;\; \kappa(x) = \frac{4F(\gamma_1(x)) - (\gamma_1'')^2 + 2\gamma_1'\gamma_1'''}{4(\gamma_1'(x))^2}.$$

Dixmier case:

$$\gamma_1 = -x, \;\; c_0 = -h, \;\; c_1 = c_2 = 0,$$

$$\chi_0 = \frac{\sqrt{z^3 - h}}{x + z} - (x^3 + h), \;\; \chi_1 = \frac{1}{x + z}.$$

Example: $g = 2$ The equations on γ_1, γ_2 have the form

$$4\gamma_1'^4\gamma_2'^2+\gamma_2'^2(c_0+c_3\gamma_1^3+c_4\gamma_1^4+\gamma_1^5-\gamma_2^2\gamma_1''^2+\gamma_1^2(c_2-\gamma_1''^2)+\gamma_1(c_1+2\gamma_2\gamma_1''^2))$$

$$+2(\gamma_2-\gamma_1)\gamma_1'^3\gamma_2'\gamma_2''+2(\gamma_1-\gamma_2)\gamma_1'\gamma_2'^2((\gamma_1-\gamma_2)\gamma_1'''-\gamma_2'\gamma_1'')$$

$$-\gamma_1'^2(c_0+c_3\gamma_2^3+c_4\gamma_2^4+\gamma_2^5$$

$$+4\gamma_2'^4-\gamma_1^2\gamma_2''^2+6\gamma_1\gamma_2'^2(\gamma_1''+\gamma_2'')+2\gamma_1^2\gamma_2'\gamma_2'''+\gamma_2^2(c_2-\gamma_2''^2+2\gamma_2'\gamma_2''')$$

$$+\gamma_2(c_1+2\gamma_1\gamma_2''^2-6\gamma_2'^2(\gamma_1''+\gamma_2'')-4\gamma_1\gamma_2'\gamma_2'''))=0.$$

Partial solution:

$$\gamma_1=-\frac{3}{2}(1+i\sqrt{3})x,\quad \gamma_2=-\frac{3}{2}(1-i\sqrt{3})x,\ w^2=z^5+27hz^2+81$$

$$L^{\sharp}=(\partial_x^2+x^3+h)^2+6x.$$

Theorem (M.) *There is a commutative ring of differential operators of rank 2 with polynomial coefficients which isomorphic to the ring of rational functions with pole in* $q = \infty$ *on spectral curve given by the equation*

$$w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \dots + c_0.$$

This ring contains operator

$$L^\sharp = (\partial_x^2 + x^3 + h)^2 + g(g+1)x.$$

As a corollary from the Theorem after changing

$$x \leftrightarrow \partial_x$$

we obtain the operator

$$(\partial_x^3 + x^2 + h)^2 + g(g+1)\partial_x.$$

of rank 3.

$$L^\vee = \partial_x^2 + x^3 + h.$$

Examples.

a) $g = 2:\ \ F_2(z) = z^5 + 27hz^2 + 81,$

$$L^\sharp_{10} = (L^\vee)^5 + \frac{15x}{2}(L^\vee)^3 + (L^\vee)^3\frac{15x}{2} + 45x^2L^\vee + L^\vee 45x^2.$$

b) $g = 3:\ \ F_3 = z^7 + 594hz^4 - 2025z^2 + 91125h^2z,$

$$L^\sharp_{14} = (L^\vee)^7 + 21x(L^\vee)^5 + (L^\vee)^5 21x + \frac{945x^2}{2}(L^\vee)^3 + (L^\vee)^3\frac{945x^2}{2} - 2709(L^\vee)^2$$

$$-(L^\vee)^2 2709 + \left(\frac{5085h}{2} + \frac{12915x^3}{2}\right)L^\vee + L^\vee\left(\frac{5085h}{2} + \frac{12915x^3}{2}\right) - 486x.$$

$$\chi_0 = -\frac{Q''/2}{Q} + \frac{\sqrt{F_g(z)}}{Q} - (x^3 + h), \;\; \chi_1 = \frac{Q'}{Q},$$

$$Q = (z - \gamma_1(x)) \ldots (z - \gamma_g(x)),$$

$$4F_g(z) + 4(gx + gx^2 - z)Q^2 + 4(x^3 + h)(Q')^2 - (Q'')^2 + 2Q'Q^{(3)}$$

$$-2Q(6x^2Q' + 4(x^3 + h)Q'' + Q^{(4)}) = 0.$$