MINIMAL TRIANGULATIONS OF PROJECTIVE PLANES

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We discuss a problem arising from the following result:

Теорема 3 (Brehm, Kühnel [1]). *Let M^d be a combinatorial manifold with n vertices. Then if $n < \lceil 3d/2 \rceil + 3$, then M is PL homeomorphic to a sphere, and if $n = 3d/2 + 3$, then either M is PL homeomorphic to a sphere, either M is a manifold "like a projective plane", i.e., manifolds with a Morse function with exactly 3 critical points (in particular, the latter manifolds exist only in dimensions $n = 2, 4, 8, 16$) (cf. [2]).*

In the smooth and topological cases, corresponding projective planes are examples of manifolds "like a projective plane". It is natural to look up for minimal triangulations in terms of number of vertices among combinatorial manifolds from Theorem 3.

In dimensions $n = 2$ and $n = 4$ these combinatorial manifolds are unique and are the minimal triangulations of the real and complex projective planes, respectively. In 1992 Brehm and Kühnel [3] constructed three PL isomorphic combinatorial manifolds corresponding to the case of $n = 8$, but they did not manage to prove the homeomorphism between these manifolds and the quaternionic projective plane. (There are still no explicit examples in the case $n = 16$.)

It follows from the work of Eells and Kuiper [2] that these manifolds are PL homeomorphic iff their Pontryagin numbers coincide. In the case of $n = 8$ it is sufficient to compute the number p_1^2, as p_2 is then computed from the classic Hirzebruch formula. In 2004 Gaifullin [4] constructed a purely combinatorial algorithm for computing the first Pontryagin class of a combinatorial manifold. Realizing this algorithm in the general case, we computed the first Pontryagin class of the three Brehm–Kühnel combinatorial manifolds. The corresponding Pontryagin numbers coincided with those of the quaternionic projective plane, thus these manifolds are PL homeomorphic to $\mathbb{H}P^2$.

Список литературы

[1] Brehm U. and Kühnel W.: Combinatorial manifolds with few vertices. Topology 26 (1987), 465–473.

[2] Eells J. and Kuiper N. H.: Manifolds which are like projective planes. Publ. Math. Inst. Hautes Etud. Sci. 14 (1962), 181–222.

[3] Brehm U. and Kühnel W.: 15-vertex triangulations of 8-manifolds. Math. Ann. 294 (1992), 167–193.

[4] Gaifullin A. A.: Local formulae for combinatorial Pontryagin classes. Izv. Math. 68 (2004), no. 5, 861–910.

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