Reductions of Kinetic Equations to Finite Component Systems

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Vlasov (Collisionless Boltzmann) Kinetic Equation

$$\lambda_t + p\lambda_x - \lambda_p u_x = 0,$$

where (λ is a distribution function, x is a space coordinate, t is a time variable, p is a momentum)

$$u = \int_{-\infty}^{\infty} \lambda dp.$$

Following the approach developed in plasma physics, one can introduce an infinite set of moments

$$A^k = \int_{-\infty}^{\infty} p^k \lambda dp.$$

The Benney hydrodynamic chain is

$$A_t^k + A_x^{k+1} + kA^{k-1}A_x^0 = 0, \quad k = 0, 1, 2, \dots$$

Kinetic Equation Describing High Frequency Waves in Electron Plasma

$$\lambda_t + p\lambda_x - E\lambda_p = 0,$$

where

$$E_x = 1 - \int_{-\infty}^{\infty} \lambda dp, \qquad E_t = \int_{-\infty}^{\infty} p\lambda dp.$$

Introducing moments of the distribution function

$$A^k = \int_{-\infty}^{\infty} p^k \lambda dp,$$

the collisionless kinetic equation can be re-written as the nonlocal chain

$$A^k_t + A^{k+1}_x + kA^{k-1}E = 0, \quad k = 0, 1, 2, \ldots,$$

where

$$dE = (1 - A^0)dx + A^1 dt$$

Waterbag Reduction

$$A^k = \frac{1}{k+1}\sum_{n=1}^{N}\epsilon_n (a^n)^{k+1}, \quad \sum_{n=1}^{N}\epsilon_n = 0,$$

where ϵ_i are arbitrary constants.
Then Benney hydrodynamic chain reduces to the form

$$a^i_t + \left(\frac{(a^i)^2}{2} + \sum_{n=1}^{N}\epsilon_n a^n\right)_x = 0,$$

where N is an arbitrary positive integer.
Nonlocal chain reduces to the nonlocal N component system

$$a^i_t + a^i a^i_x + E = 0, \qquad dE = (1 - A^0)dx + A^1 dt.$$

Local Representation

The nonlocal chain

$$A_t^k + A_x^{k+1} + kA^{k-1}E = 0, \quad k = 0, 1, 2, \ldots,$$

where

$$dE = (1 - A^0)dx + A^1 dt$$

can be written in the evolution form

$$E_t = A^1, \quad A_t^1 + A_x^2 + E(1 - E_x) = 0,$$

$$A_t^k + A_x^{k+1} + kA^{k-1}E = 0, \quad k = 2, 3, \ldots,$$

where $A^0 = 1 - E_x$. In such a form this non-hydrodynamic chain has just one local conservation law of Energy

$$(A^2 + E^2)_t + A_x^3 = 0.$$

Local Representation

The nonlocal system

$$a_t^k + a^k a_x^k + E = 0, \qquad dE = (1 - A^0)dx + A^1 dt$$

transforms to N component evolution system $(k = 2, 3, \dots, N)$

$$a_t^k + a^k a_x^k + E = 0,$$

$$E_t = \frac{1}{2}\sum_{n=2}^{N} \epsilon_n (a^n)^2 + \frac{1}{2\epsilon_1}\left(E_x + \sum_{n=2}^{N} \epsilon_n a^n - 1\right)^2,$$

where

$$a^1 = \frac{1}{\epsilon_1}\left(1 - \sum_{n=2}^{N} \epsilon_n a^n - E_x\right).$$

Hamiltonian Structure

The waterbag reduction can be written in the Hamiltonian form

$$E_t = -\sum_{n=2}^{N} \frac{\delta \mathbf{H}}{\delta a^n}, \quad a^i_t = \frac{1}{\epsilon_i}\partial_x \frac{\delta \mathbf{H}}{\delta a^i} + \frac{\delta \mathbf{H}}{\delta E}, \;\; i = 2, 3, ..., N,$$

which is determined by the local Hamiltonian

$$\mathbf{H} = -\frac{1}{2}\int \left[E^2 + \frac{1}{3}\sum_{n=2}^{N} \epsilon_n (a^n)^3 + \frac{1}{3\epsilon_1^2}\left(E_x + \sum_{n=2}^{N} \epsilon_n a^n - 1 \right)^3 \right] dx,$$

the momentum

$$\mathbf{P} = \int \left[\frac{1}{2}\sum_{n=2}^{N} \epsilon_n (a^n)^2 + \frac{1}{2\epsilon_1}\left(\sum_{n=2}^{N} \epsilon_n a^n \right)^2 + \frac{E_x}{\epsilon_1}\sum_{n=2}^{N} \epsilon_n a^n + \frac{E_x^2}{2\epsilon_1} \right] dx$$

and $N-2$ parametric Casimir

$$\mathbf{Q} = \int \sum_{n=2}^{N} \tilde{\epsilon}_n a^n, \qquad \sum_{n=2}^{N} \tilde{\epsilon}_n = 0.$$

Periodic Solutions

Each field variable a^k depend on a sole phase $\theta = x - \omega t$.
Then the nonlocal system

$$a^k_t + a^k a^k_x + E = 0, \qquad dE = (1 - A^0)dx + A^1 dt$$

reduces to an ODE system (ξ and β_k are arbitrary constants)

$$u'^2 = \frac{\xi}{u^2} - \delta + \frac{2u}{3}\epsilon_1 \pm \frac{2}{3u^2}\sum_{k=2}^{N}\epsilon_k(u^2 + \beta_k^2)^{3/2},$$

where

$$a^1 = u + \omega, \quad a^k = \omega \pm [u^2 + \beta_k^2]^{1/2}, \quad k = 2, 3, ..., N,$$

This is N parametric periodic solution, whose characteristic velocity is determined by

$$\omega = -\frac{1}{2\delta}\sum_{k=2}^{N}\epsilon_k\beta_k^2,$$

which follows from comparison both above expressions for E'.

Periodic Solutions

Since $E = -uu'$, its explicit expression is

$$E = -\sqrt{\xi - \delta u^2 + \frac{2u^3}{3}\epsilon_1 \pm \frac{2}{3}\sum_{k=2}^{N} \epsilon_k (u^2 + \beta_k^2)^{3/2}}.$$

Then a corresponding solution of the distribution function $\lambda(x, t, p)$ can be found due to the inverse formula

$$\lambda(x, t, p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iqp} \sum_{k=0}^{\infty} \frac{(-iq)^k}{k!} A^k dq,$$

where

$$A^k = \frac{1}{k+1}\sum_{n=2}^{N} \epsilon_n [(\omega \pm [u^2 + \beta_n^2]^{1/2})^{k+1} - (u+\omega)^{k+1}].$$

Kodama Reduction

N component Kodama reductions

$$a_t^0 + a^0 a_x^0 + E = 0, \ \ E_x = \delta + a^N, \ \ a_t^k + \frac{1}{2}\left(\sum_{m=0}^{k} a^m a^{k-m}\right)_x = 0, \ \ k = 1, \dots, N$$

possess the local Hamiltonian structure

$$a_t^0 = -\frac{\delta \mathbf{H}_{(N)}}{\delta E}, \quad E_t = \frac{\delta \mathbf{H}_{(N)}}{\delta a^0}, \quad a_t^k = \partial_x \frac{\delta \mathbf{H}_{(N)}}{\delta a^{N-k}}, \ \ k = 1, 2, \dots, N-1,$$

where the Hamiltonian is given by

$$\mathbf{H}_{(N)} = \frac{1}{2}\int \left[E^2 + (a^0)^2(\delta - E_x) - \frac{a^0}{3}\sum_{k=1}^{N-1} a^k a^{N-k} - \frac{1}{3}\sum_{n=1}^{N-1}\sum_{k=0}^{n} a^k a^{n-k} a^{N-n}\right]$$

and the Momentum is

$$\mathbf{P} = \int \left[\frac{1}{2}\sum_{k=1}^{N-1} a^k a^{N-k} + a^0 E_x\right] dx.$$

Periodic Solutions

Each field variable a^i depend on a sole phase $\theta = x - \omega t$. Then Kodama reduction

$$a^0_t + a^0 a^0_x + E = 0,\ \ E_x = \delta + a^N,\ \ a^k_t + \frac{1}{2}\left(\sum_{m=0}^{k} a^m a^{k-m}\right)_x = 0,\ \ k = 1, ..., N$$

becomes to the form of two ordinary differential equations of the first order ($u = a^0 - \omega$)

$$uu' + E = 0, \quad E' = \delta + a^N,$$

while all other functions can be found iteratively

$$a^1 = -\frac{\gamma_1}{u},\ \ a^2 = -\frac{1}{u}\left(\gamma_2 + \frac{1}{2}(a^1)^2\right), ..., \quad a^N = -\frac{1}{u}\left(\gamma_N + \frac{1}{2}\sum_{m=1}^{N-1} a^m a^{N-m}\right.$$

where γ_k are integration constants.

Periodic Solutions

Then periodic solution has the form (ξ is an arbitrary constant)

$$u'^2 = \delta + \frac{\xi}{u^2} + \frac{2}{u^2}\int u a^N(u) du$$

and can be integrated in hyperelliptic functions. If $N = 1$, then

$$u'^2 = \delta - \frac{2\gamma_1}{u} + \frac{\xi}{u^2};$$

if $N = 2$, then

$$u'^2 = \delta - \frac{2\gamma_2}{u} + \frac{\xi}{u^2} + \frac{\gamma_1^2}{u^3};$$

if $N = 3$, then

$$u'^2 = \delta - \frac{2\gamma_3}{u} + \frac{\xi}{u^2} + \frac{2\gamma_1\gamma_2}{u^3} + \frac{\gamma_1^3}{3u^5};$$

and so on.

Waterbag Reduction. Heaviside Step Function

Let us consider the ansatz

$$\lambda = \sum_{i=1}^{N-1} f_i \left[\theta(p - a^i(t,x)) - \theta(p - a^{i+1}(t,x))\right].$$

We assume that $a^i < a^{i+1}$, $f_i = \mathrm{const} > 0$. Let us suppose that

$$f_i = -\sum_{n=1}^{i} \varepsilon_n \quad (i = 1, ..., N-1), \quad f_N = \sum_{n=1}^{N} \varepsilon_n = 0.$$

Then we have

$$\lambda = -\sum_{i=1}^{N} \varepsilon_i \theta(p - a^i(t,x)).$$

Substitution to the kinetic equation

$$\lambda_t + p\lambda_x - E\lambda_p = 0, \quad E_x = 1 - \int_{-\infty}^{\infty} \lambda dp, \quad E_t = \int_{-\infty}^{\infty} p\lambda dp$$

yields N component reduction

$$a_t^i + a^i a_x^i + E = 0, \quad (i = 1, ..., N)$$

$$E_x = 1 - \sum_{n=1}^{N-1} f_n (a^{n+1} - a^n) = 1 - \sum_{n=1}^{N} \varepsilon_n a^n,$$

$$E_t = \frac{1}{2} \sum_{n=1}^{N-1} f_n \left[(a^{n+1})^2 - (a^n)^2 \right] = \frac{1}{2} \sum_{n=1}^{N} \varepsilon_n (a^n)^2,$$

which coincide with already found waterbag reduction via the moment decomposition

$$A^k = \frac{1}{k+1} \sum_{n=1}^{N} \varepsilon_n (a^n)^{k+1}, \quad \sum_{n=1}^{N} \varepsilon_n = 0.$$

Cold Plasma. Multi-flow Hydrodynamics

$$\lambda = \sum_{i=1}^{N} b_i(t,x)\delta(p - a^i(t,x))$$

Substitution to the kinetic equation

$$\lambda_t + p\lambda_x - E\lambda_p = 0, \quad E_x = 1 - \int_{-\infty}^{\infty} \lambda dp, \quad E_t = \int_{-\infty}^{\infty} p\lambda dp$$

implies

$$\sum_{i=1}^{N} \left[(b^i_t + pb^i_x)\delta(p - a^i) - b^i(a^i_t + pa^i_x + E)\delta'(p - a^i) \right] = 0.$$

Taking into account the following equalities

$$\varphi(p)\delta(p - a^i) = \varphi(a^i), \quad \varphi(p)\delta'(p - a^i) = -\varphi'(a^i),$$

Cold Plasma. Multi-flow Hydrodynamics

we obtain

$$a_t^i + a^i a_x^i + E = 0, \quad b_t^i + (a^i b^i)_x = 0, \quad (i = 1, ..., N)$$

$$E_x = 1 - \sum_{n=1}^{N} b^n, \quad E_t = \sum_{n=1}^{N} a^n b^n.$$

This reduction is determined by the so called Zakharov moment decomposition

$$A^k = \sum_{n=1}^{N} (a^n)^k b^n.$$

Generalizations

Let us consider more general ansatz

$$\lambda = \sum_{i=1}^{N} \left[c^i(t,x)\delta(p - a^i(t,x)) + e^i(t,x)\delta'(p - a^i(t,x)) \right].$$

Substitution to the kinetic equation

$$\lambda_t + p\lambda_x - E\lambda_p = 0, \quad E_x = 1 - \int\limits_{-\infty}^{\infty} \lambda dp, \quad E_t = \int\limits_{-\infty}^{\infty} p\lambda dp$$

implies

$$\sum_{i=1}^{N} \Big[(c^i_t + pc^i_x)\delta(p - a^i) + \big(e^i_t - c^i a^i_t + p(e^i_x - c^i a^i_x) - Ec^i\big)\delta'(p - a^i) \\ - e^i(a^i_t + pa^i_x + E)\delta''(p - a^i) \Big] = 0.$$

Then we obtain

$$
\begin{aligned}
& c^i_t + (a^i c^i)_x - e^i_x = 0, \\
& c^i(a^i_t + a^i a^i_x + E) - (e^i_t + a^i e^i_x + 2e^i a^i_x) = 0, \\
& (a^i c^i - e^i)(a^i_t + a^i a^i_x + E) - a^i(e^i_t + a^i e^i_x + 2e^i a^i_x) = 0.
\end{aligned}
$$

Introducing new functions b^i such that $e^i = -(b^i)^2/2$ we come back to the system

$$
\begin{gathered}
a^i_t + a^i a^i_x + E = 0, \quad b^i_t + (a^i b^i)_x = 0, \quad c^i_t + \left(a^i c^i + \frac{(b^i)^2}{2}\right)_x = 0, \\
E_x = 1 - \sum_{n=1}^{N} c^n, \quad E_t = \sum_{n=1}^{N}\left(a^n c^n + \frac{(b^n)^2}{2}\right)
\end{gathered}
$$

is associated with the moment decomposition

$$
A^k = \sum_{n=1}^{N}\left[(a^n)^k c^n + \frac{k}{2}(a^n)^{k-1}(b^n)^2\right].
$$

Next $4N$ component nonlocal system

$$\begin{aligned} a^i_t + a^i a^i_x + E &= 0, \quad b^i_t + (a^i b^i)_x = 0, \quad c^i_t + \left(a^i c^i + \frac{(b^i)^2}{2}\right)_x = 0, \\ g^i_t + \left(a^i g^i + b^i c^i\right)_x &= 0, \quad dE = (1 - A^0)dx + A^1 dt \end{aligned}$$

is associated with the moment decomposition

$$A^k = \sum_{n=1}^{N} (a^n)^k g^n - k \sum_{n=1}^{N} (a^n)^{k-1} b^n c^n + \frac{k(k-1)}{6} \sum_{n=1}^{N} (a^n)^{k-2} (b^n)^3.$$

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