A SYSTEMATIC CONSTRUCTION OF NONLOCALLY RELATED SYSTEMS OF PDEs

1. **Construction of nonlocally related PDE systems**
   --Use of CL to obtain nonlocally related system (potential system)
   --Use of n CLs to obtain up to 2^n nonlocally related systems
   --How to find nonlocally related subsystems
   --Tree of nonlocally related systems

2. **Examples**
   --Nonlinear wave equations
   --Nonlinear telegraph equations
   --Planar gas dynamics equations

Limitation: **A given PDE system as it stands does not have a useful local symmetry or useful local conservation law**

Aim: **To extend the existing methods to systems that are nonlocally related but equivalent to a given system of PDEs**

How to do this systematically?

A natural way to do this is through the use of conservation laws (CLs)

CONSTRUCTION OF NONLOCALLY RELATED SYSTEMS THROUGH CLs

Given any local CL

$$\mathrm{D}_x X(x,t,u,\partial u,\ldots,\partial^r u)+\mathrm{D}_t T(x,t,u,\partial u,\ldots,\partial^r u)=0$$

of

$$R[u]=R(x,t,u,\partial u,\ldots,\partial^k u)=0, \qquad (1)$$

one can form an **equivalent** augmented *potential system P*

$$\begin{aligned}
\frac{\partial v}{\partial t} &= X(x,t,u,\partial u,\ldots,\partial^r u),\\
\frac{\partial v}{\partial x} &= -T(x,t,u,\partial u,\ldots,\partial^r u),\\
R(x,t,u,&\partial u,\ldots,\partial^k u)=0
\end{aligned}$$

If (u,v) solves potential system P then u solves $R[u]=0$.

Conversely, if u solves $R[u]=0$, then there exists a solution (u,v) of the potential system P due to integrability conditions $v_{xt}=v_{tx}$ being satisfied from CL.

But the equivalence relationship is *nonlocal and non-invertible* since for any u solving $R[u] = 0$, if (u,v) solves the potential system P, then so does $(u,v+C)$ for any constant C.

A symmetry (CL) of $R[u] = 0$ yields a symmetry (CL) of the potential system P.

Conversely, a symmetry (CL) of the potential system P yields a symmetry (CL) of $R[u] = 0$.

Suppose

$$\mathrm{X} = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \omega(x,t,u,v)\frac{\partial}{\partial u} + \phi(x,t,u,v)\frac{\partial}{\partial v}$$

is a point symmetry of the equivalent potential system *P*. Then X yields a **nonlocal symmetry** of the given PDE (1) iff

$$(\xi_v)^2 + (\tau_v)^2 + (\omega_v)^2 \not\equiv 0.$$

Hence through a CL of (1), a nonlocal symmetry of (1) can be obtained through a point (local) symmetry of the related potential system *P*. [Converse is also true!]

Use of *n* CLs to obtain up to 2^n nonlocally related systems

Now suppose, there are n multipliers $\{\Lambda_i(x,t,U,\partial U,\ldots,\partial^q U)\}$ yielding n independent CLs of $R[u] = 0$.

Let v^i be the potential variable $\leftrightarrow$ multiplier $\Lambda_i[U]$

Then we obtain n **singlet** potential systems
$P^i, \quad i=1,\ldots,n$

Moreover, we can consider the potential systems

in **couplets** $\{P^i, P^j\}_{i,j=1}^n$ with two potential variables

in **triplets** $\{P^i, P^j, P^k\}_{i,j,k=1}^n$ with three potential variables

$,\ldots,$

in an ***n*-plet** $\{P^1,\ldots,P^n\}$ with n potential variables

Hence from n CLs, we obtain $2^n - 1$ *distinct potential systems!*

Starting from *any* potential system, we can continue the process and if it has N "local" CLs, we can obtain up to $2^N - 1$ further distinct potential systems. One can tell in advance whether or not one obtains further potential systems.

In particular, one can show that if the multipliers depend only on the local variables (x, t, u) **then no new potential system is obtained.**

Any one of these potential systems could yield new nonlocal symmetries or new nonlocal CLs for any one of the other potential systems or the "given" PDE system.

Nonlocally related subsystems

Suppose we are given a system of PDEs $S\{x,t,u^1,\ldots,u^M\}=0$ with the indicated M dependent variables.

A **subsystem** excluding one of the dependent variables, say u^M, $\underline{S}\{x,t,u^1,\ldots,u^{M-1}\}=0$ is *nonlocally related* to the given system $S\{x,t,u^1,\ldots,u^M\}=0$ if u^M cannot be directly expressed from the equations of $S\{x,t,u^1,\ldots,u^M\}=0$ in terms of $x,t,$ the remaining dependent variables $u^1,\ldots,u^{M-1},$
and their derivatives.

Subsystems for consideration can arise following an interchange of dependent and independent variables of the given system

$$S\{x,t,u^1,\ldots,u^M\}=0$$

Tree of Nonlocally Related Systems

Consequently, for a given system one obtains a tree of nonlocally related (but equivalent) systems arising from CLs and subsystems.

Each system in such an extended tree is equivalent in the sense that the solution set for any system in a tree can be found from the solution set for any other system in the tree.

Due to the equivalence of the solution sets and the nonlocal relationship, it follows that any coordinate-independent method of analysis (quantitative, analytical, numerical, perturbation, etc.) when applied to any system in the tree may yield simpler computations and/or results that cannot be obtained when the method is directly applied to the given system.

Note also that the “given” system could be any system in a tree!!

EXAMPLES

1. Nonlinear wave equation

Suppose the given PDE is the nonlinear wave equation

$$\mathbf{U}\{x,t,u\}=0: \quad u_{tt}=(c^2(u)u_x)_x$$

Directly, we obtain the singlet potential system (multiplier is 1)

$$\mathbf{UV}\{x,t,u,v\}=0: \quad \begin{cases} v_x-u_t=0, \\ v_t-c^2(u)u_x=0 \end{cases}$$

By the invertible point transformation (hodograph)
$x=x(u,v), t=t(u,v)$,
the **UV** potential system becomes

$$\mathbf{XT}\{x,t,u,v\}=0: \quad \begin{cases} x_v-t_u=0, \\ x_u-c^2(u)t_v=0 \end{cases}$$

One can show that there are only three more multipliers of the form $\Lambda(x,t,u) = xt, x, t$ that yield CLs for **U** for an *arbitrary* wave speed $c(u)$.

This yields three more singlet potential systems given by

$$\mathbf{UA}\{x,t,u,a\} = 0: \quad \begin{cases} a_x - x[tu_t - u] = 0, \\ a_t - t[xc^2(u)u_x - \int c^2(u)du] = 0 \end{cases}$$

$$\mathbf{UB}\{x,t,u,b\} = 0: \quad \begin{cases} b_x - xu_t = 0, \\ b_t - [xc^2(u)u_x - \int c^2(u)du] = 0 \end{cases}$$

$$\mathbf{UW}\{x,t,u,w\} = 0: \quad \begin{cases} w_x - [tu_t - u] = 0, \\ w_t - t\int c^2(u)du = 0 \end{cases}$$

Nonlocally related subsystems arise from **UV** through **XT**:

$$\mathbf{T}\{u,v,t\} \equiv \mathbf{L}\{u,v,t\} = 0: \quad t_{vv} - c^{-2}(u)t_{uu} = 0$$

$$\mathbf{X}\{u,v,x\} = 0: \quad x_{vv} - (c^{-2}(u)x_u)_u = 0$$

One can show that the symmetry classifications of these two PDEs are "equivalent". Hence we concentrate on **T.**

One can show that there are only four multipliers of the form

$$\Lambda(u,v,t) = c^2(u), uc^2(u), vc^2(u), uvc^2(u)$$

that yield CLs for **T** for an arbitrary wave speed $c(u)$.

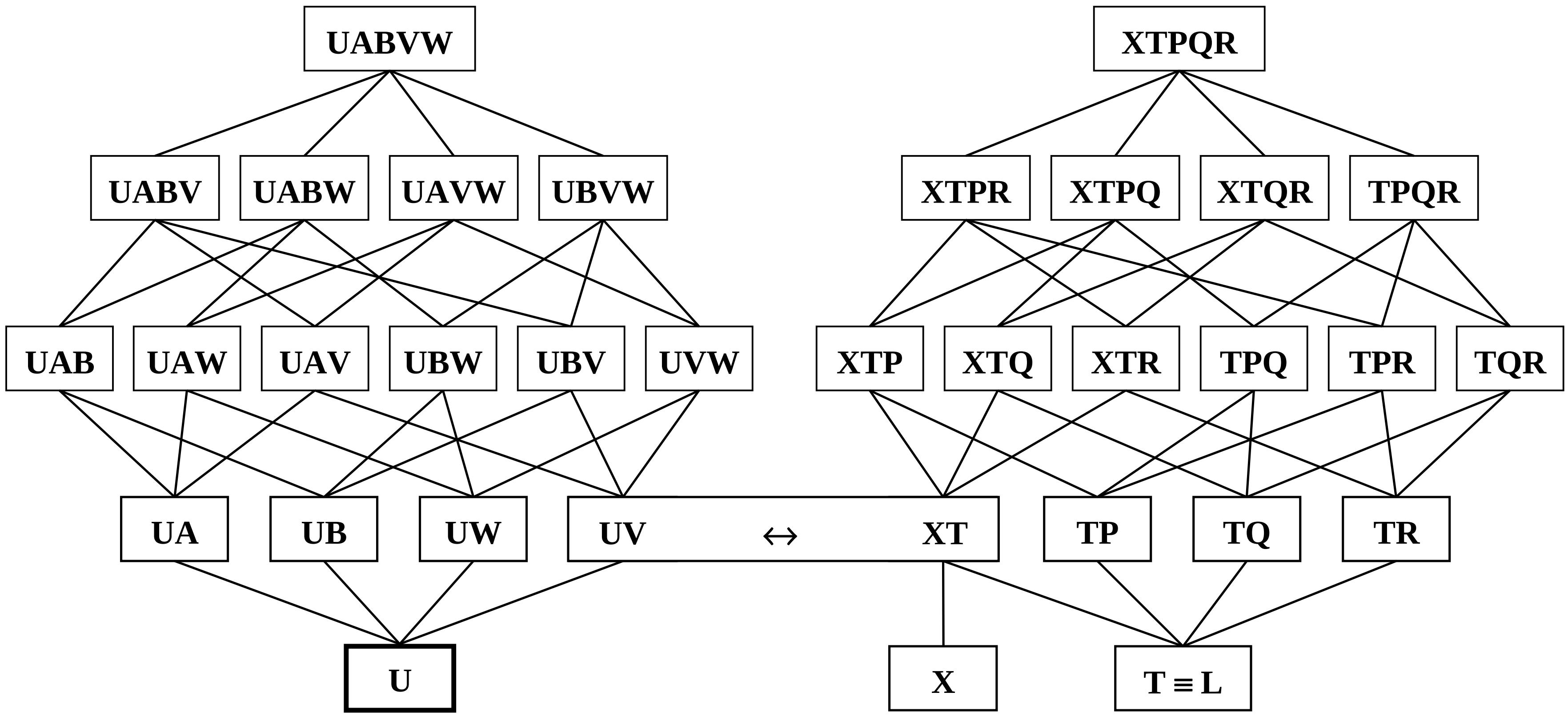
The resulting new singlet potential systems include

$$\mathbf{TP}\{u,v,t,p\}=0: \quad \begin{cases} p_v-(ut_u-t)=0, \\ p_u-uc^2(u)t_v=0 \end{cases}$$

$$\mathbf{TQ}\{u,v,t,q\}=0: \quad \begin{cases} q_v-vt_u=0, \\ q_u+c^2(u)(t-vt_v)=0 \end{cases}$$

$$\mathbf{TR}\{u,v,t,r\}=0: \quad \begin{cases} r_v-v(ut_u-t)=0, \\ r_u-uc^2(u)[vt_v-t]=0 \end{cases}$$

Consequently, one obtains the following **(far from exhaustive)** tree of nonlocally related systems for the nonlinear wave equation **U** for an *arbitrary* wave speed $c(u)$.


UABVW
XTPQR
UABV
UABW
UAVW
UBVW
XTPR
XTPQ
XTQR
TPQR
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XT
TP
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U
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T ≡ L

- Point symmetry classification of the nonlinear wave equation **U** was given in Ames, Lohner & Adams (1981)

- Point symmetry classifications of potential system **XT** and subsystem **T** were given in B & Kumei (1987)

- Partial point symmetry classifications of the potential systems **TP** and **TQ** can be adapted from results presented in Ma (1990).

- Complete point symmetry classifications of potential systems **UA, UB, UW, TP, TQ** are given in B & Cheviakov (2007). Many nonlocal symmetries for the nonlinear wave equation are found from each of these nonlocally related systems in terms of specific forms of the nonlinear wave speed $c(u)$. In particular, the following new nonlocal symmetries for the nonlinear wave equation **U** were found:

For potential system **UB**, setting $F(u)=\int c^2(u)\,du,$ one finds that if $F(u)$ satisfies the ODE

$$\frac{F''(u)}{F'(u)^2}=\frac{4F(u)+2C_1}{(F(u)+C_2)^2+C_3},$$

with arbitrary constants $C_1, C_2, C_3,$ then the potential system **UB** has the point symmetry

$$\begin{aligned}\mathrm{X}=(F(u)+C_1)x\frac{\partial}{\partial x}+b\frac{\partial}{\partial t}+\frac{(F(u)+C_2)^2+C_3}{F'(u)}\frac{\partial}{\partial u}\\ +(2C_2b-(C_2{}^2+C_3)t)\frac{\partial}{\partial b}\end{aligned}$$

that is a nonlocal symmetry of the nonlinear wave equation **U**.

For potential system **UW** if $c(u)$ satisfies the ODE

$$\frac{c'(u)}{c(u)} = -\frac{2u + C_1}{u^2 + C_2},$$

with arbitrary constants $C_1, C_2,$ then it has the point symmetry

$$\mathrm{X} = w\frac{\partial}{\partial x} + (u + C_1)t\frac{\partial}{\partial t} + (u^2 + C_2)\frac{\partial}{\partial u} - C_2 x\frac{\partial}{\partial w},$$

that is a nonlocal symmetry of the nonlinear wave equation **U**.

For potential system **TP**, if

$$c(u) = u^{-2} e^{1/u},$$

it has the point symmetries

$$\mathrm{X}_1 = (pu - 2tv(u+1))\frac{\partial}{\partial t} - 2u^2 v\frac{\partial}{\partial u} + (u^2 + e^{2/u})\frac{\partial}{\partial v} + tu^{-1}e^{2/u}\frac{\partial}{\partial p},$$
$$\mathrm{X}_2 = t(u+1))\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u} - v\frac{\partial}{\partial v},$$

that are both nonlocal symmetries of the nonlinear wave equation **U**

For potential system **TR,** new nonlocal symmetries are found for **U** from its point symmetries when

$$c(u) = u^{-4/3}.$$

Table 2
Cases in which nonlocal symmetries of the nonlinear wave equation **U** (1.1) arise

System	Nonlocal variable(s)	Condition on $c(u)$	Symmetries; remarks
UA (2.12)	a	No special cases	Nonlocal symmetries do not arise.
UB (2.13)	b	$c(u)=u^{-2/3}$	Linearizable by a point transformation.
		$\frac{F''(u)}{(F'(u))^2}=\frac{4F(u)+C_1}{(F(u)+C_2)^2+C_3}$ $(F(u)=\int c^2(u)\,du,\ C_1, C_2, C_3=\mathrm{const})$	One nonlocal symmetry.
UW (2.14)	w	$c(u)=u^{-2}$	Linearizable by a point transformation.
		$\frac{c'(u)}{c(u)}=-\frac{2u+C_1}{u^2+C_2}\ (C_1, C_2=\mathrm{const})$	One nonlocal symmetry.
XT (2.5) or **UV** (2.3)	v	$[\frac{c'(u)}{c^3(u)}(\frac{c(u)}{c'(u)})'']'=0$	One or two nonlocal symmetries; adapted from [4].
TP (2.22)	v, p	$\frac{-(2uc^2+u^2cc')c'''+2u^2c(c'')^2}{c^3(uc'+2c)^2}+\frac{-(4c^2+u^2(c')^2-8ucc')c''+6(c-uc')(c')^2}{c^3(uc'+2c)^2}=\lambda^2,$ $\lambda=\mathrm{const}$	One or two nonlocal symmetries; partially adapted from [6].
		$c(u)=u^{-2}$	Infinite number of nonlocal symmetries; there exists a point mapping into a system with constant coefficients.
TQ (2.23)	v, q	$c(u)=u^{-2/3}$; $c(u)=u^{-2}$	Two nonlocal symmetries; partially adapted from [6].
TR (2.24)	v, r	$\frac{ucc''+(c-uc')c'}{(uc'+2c)^2}=\gamma^2=\mathrm{const}$	Two nonlocal symmetries.
L (2.15)	v	$(\alpha'+H\alpha)'=\sigma^2\alpha c^2(u),\ \sigma=\mathrm{const}$ $(H=c'(u)/c(u),\ \alpha^2=(H^2-2H')^{-1})$	One or two nonlocal symmetries; adapted from [4].
		$c(u)=u^{-2}$	Infinite number of nonlocal symmetries; there exists an invertible mapping into a system with constant coefficients [9].
X (2.16)	v	$\frac{(-2cc''+5(c')^2)c^2c''''+3c^3(c''')^2+16c^2(c'')^3}{c^3(2cc''-5(c')^2)^2}+\frac{-24c^2c'c''c'''+12c(c'c'')^2-10(c')^4c''}{c^3(2cc''-5(c')^2)^2}=\sigma^2,$ $\sigma=\mathrm{const}$	One or two nonlocal symmetries; partially adapted from [6].

2. Nonlinear Telegraph Equation

Suppose the given PDE is the nonlinear telegraph (NLT) equation

$$\mathbf{U}\{x,t,u\}=0: \quad u_{tt}-(F(u)u_x)_x-(G(u))_x=0$$

Case (a) For *arbitrary* $F(u), G(u),$ one obtains two singlet potential systems

$$\mathbf{UV}_1\{x,t,u,v_1\}=0: \quad \begin{cases} v_{1x}-u_t=0, \\ v_{1t}-(F(u)u_x+G(u))=0 \end{cases}$$

$$\mathbf{UV}_2\{x,t,u,v_2\}=0: \quad \begin{cases} v_{2x}-(tu_t-u)=0, \\ v_{2t}-t(F(u)u_x+G(u))=0 \end{cases}$$

Case (b) For *arbitrary* $G(u),\ F(u)=G'(u),$ one obtains two more singlet potential systems

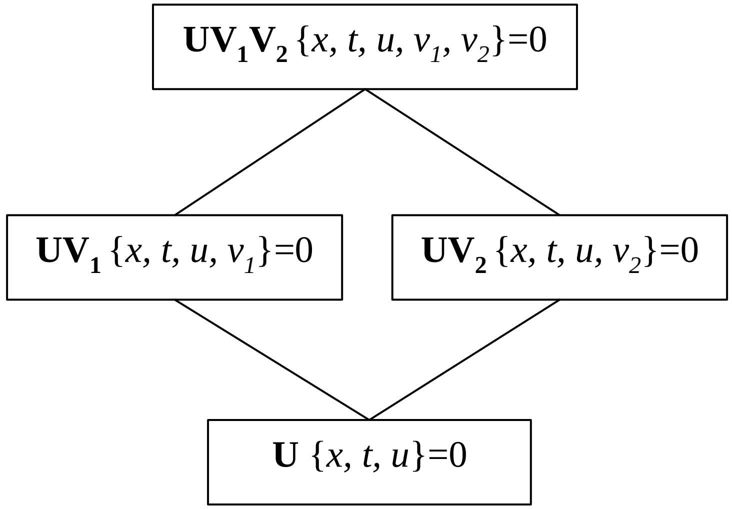
$$\mathbf{UB}_3\{x,t,u,b_3\}=0: \quad \begin{cases} b_{3x}-e^x u_t=0, \\ b_{3t}-e^x F(u)u_x=0 \end{cases}$$

$$\mathbf{UB}_4\{x,t,u,b_4\}=0: \quad \begin{cases} b_{4x}-e^x(tu_t-u)=0, \\ b_{4t}-te^x F(u)u_x=0 \end{cases}$$

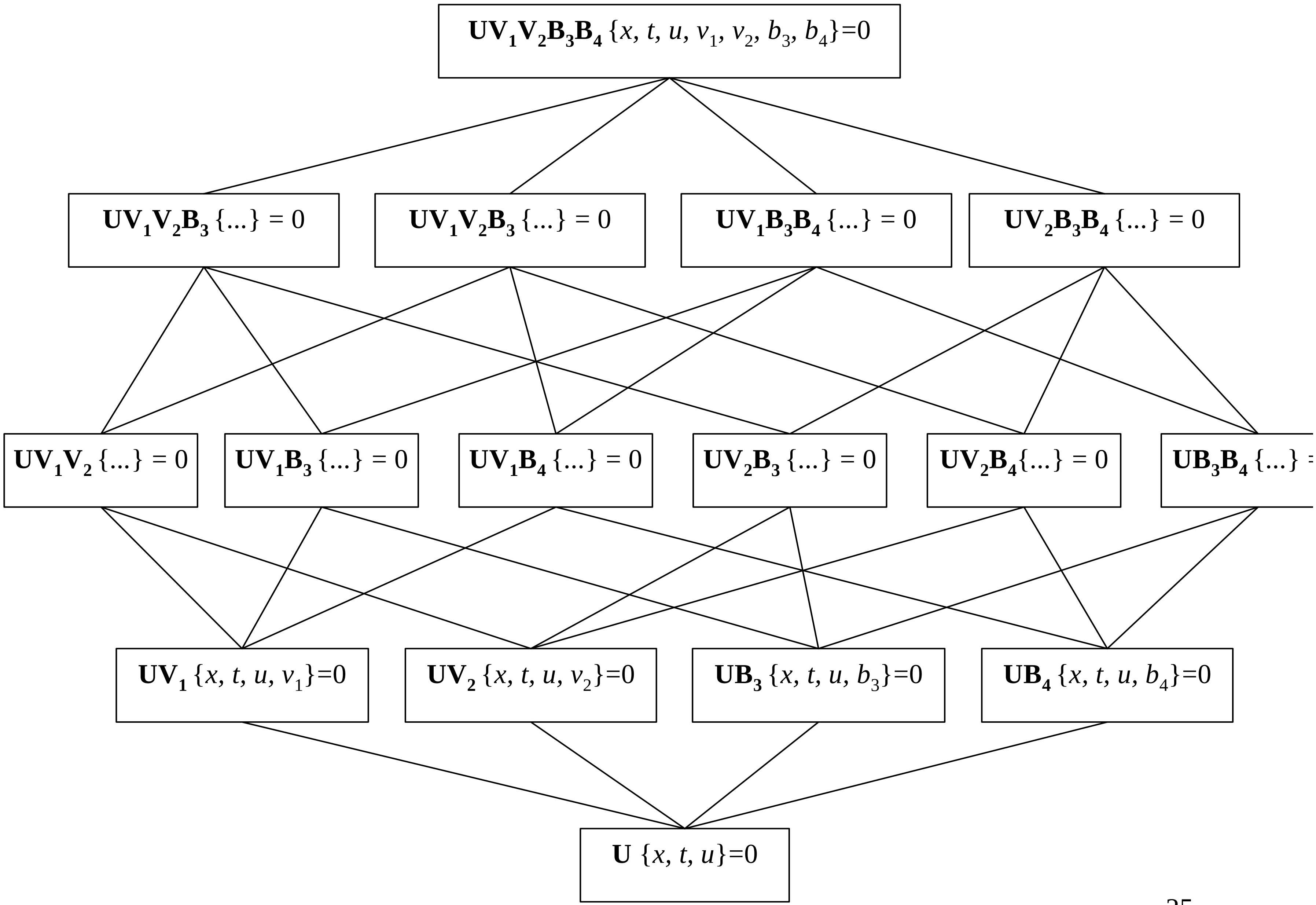
Case (c) $F(u)$ arbitrary, $G(u) = u$: In addition to the first two singlet potential systems, there are two more:

$$\mathbf{UC}_3\{x,t,u,c_3\} = 0: \quad \begin{cases} c_{3x} - ((x - \tfrac{1}{2}t^2)u_t + tu) = 0, \\ c_{3t} - (x - \tfrac{1}{2}t^2)(F(u)u_x + u) + \int F(u)d \end{cases}$$

$$\mathbf{UC}_4\{x,t,u,c_4\} = 0: \quad \begin{cases} c_{4x} + (\tfrac{1}{6}t^3 - tx)u_t + (x - \tfrac{1}{2}t^2)u = 0, \\ c_{4t} + (\tfrac{1}{6}t^3 - tx)(F(u)u_x + u) + t\int F(u) \end{cases}$$



Tree of nonlocally related systems for NLT eqn for arbitrary $F(u)$, $G(u)$



Tree of nonlocally related systems for NLT eqn for arbitrary $G(u)$, $F = G'$

TABLE IV. Symmetries of the potential NLT systems for case for case (b): $F(u)=(\alpha+1)u^{\alpha}$, $G(u)=u^{\alpha+1}(\alpha\neq 0,-1)$.

System	$F(u)$	$G(u)$	Symmetries
$\mathbf{UV_1V_2B_3B_4}$, $\mathbf{UV_1V_2B_3}$, $\mathbf{UV_1V_2B_4}$, $\mathbf{UV_1B_3B_4}$, $\mathbf{UV_2B_3B_4}$, $\mathbf{UV_1V_2}$, $\mathbf{UV_1B_3}$, $\mathbf{UV_1B_4}$, $\mathbf{UV_2B_3}$, $\mathbf{UV_2B_4}$, $\mathbf{UB_3B_4}$, $\mathbf{UV_1}$, $\mathbf{UV_2}$, $\mathbf{UB_3}$, $\mathbf{UB_4}$, $\mathbf{U}$	$(\alpha+1)u^{\alpha}$	$u^{\alpha+1}$	$Y_1=-\frac{\alpha}{2}t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u}+v_2\frac{\partial}{\partial v_2}+\frac{\alpha+2}{2}v_1\frac{\partial}{\partial v_1}+\frac{\alpha+2}{2}b_3\frac{\partial}{\partial b_3}+b_4\frac{\partial}{\partial b_4}$, $Y_2=\frac{\partial}{\partial x}+b_3\frac{\partial}{\partial b_3}+b_4\frac{\partial}{\partial b_4}$, $Y_3=\frac{\partial}{\partial t}+b_3\frac{\partial}{\partial b_4}+v_1\frac{\partial}{\partial v_2}$, $Y_4=\frac{\partial}{\partial v_1}$, $Y_5=\frac{\partial}{\partial v_2}$, $Y_6=\frac{\partial}{\partial b_3}$, $Y_7=\frac{\partial}{\partial b_4}$
	$-3u^{-4}$	u^{-3}	$Y_8=t^2\frac{\partial}{\partial t}+tu\frac{\partial}{\partial u}-v_2\frac{\partial}{\partial v_1}-b_4\frac{\partial}{\partial b_3}$
$\mathbf{UV_1V_2}$	$3u^2$	u^3	$Y_9=3v_1\frac{\partial}{\partial x}+(tv_1-v_2+3u)\frac{\partial}{\partial t}-uv_1\frac{\partial}{\partial u}-v_1^2\frac{\partial}{\partial v_1}-v_1v_2\frac{\partial}{\partial v_2}$

TABLE VI. Nonlocal conservation laws of (4.1).

Case	System	Subcase	Multipliers	Fluxes
(a)	UV_1	$\beta=-1$	$\Lambda_1=x+\frac{v_1^2}{2}+\frac{u^{\alpha+2}}{\alpha+2},\Lambda_2=uv_1$	$X=-\left(\frac{u^{\alpha+2}}{\alpha+2}+\frac{v_1^2}{6}+x\right)v_1,$ $T=\left(\frac{u^{\alpha+2}}{(\alpha+2)(\alpha+3)}+\frac{v_1^2}{2}+x\right)u.$
$F(u)=u^\alpha$			$\Lambda_1=v_1,\Lambda_2=u.$	$X=-\frac{u^{\alpha+2}}{\alpha+2}-\frac{v_1^2}{2},$ $T=uv_1-t.$
$G(u)=u^\beta$		$\alpha=-1$ $\beta=-1$	$\Lambda_1=\frac{v_1^3}{3}+2(x+u)v_1+t,$ $\Lambda_2=(v_1^2+u+2x)u.$	$X=-\frac{v_1^4}{12}-(x+u)v_1^2-tv_1-\frac{u^2}{2}-2xu,$ $T=\left(u+\frac{v_1^2}{3}\right)uv_1+2xuv_1+t(u-2x).$
			$\Lambda_1=v_1^4/12+(u+x)v_1^2+tv_1+2xu+x^2+\frac{u^2}{2},$ $\Lambda_2=\left(\frac{v_1^3}{3}+t+uv_1+2xv_1\right)u.$	$X=-\frac{v_1^5}{60}-\frac{(x+u)v_1^3}{3}-\frac{(tv_1+u^2)v_1}{2}-(2u+x)xv_1-tu,$ $T=-\frac{t^2}{2}+\left(\frac{u}{3}+v_1^2+2x\right)\frac{u^2}{2}+\frac{uv_1^4}{12}+(xv_1+t)uv_1+x^2u.$
	UV_2	$\beta=-1$	$\Lambda_1=-\frac{v_2}{t^2},\Lambda_2=\frac{u}{t}.$	$X=-\frac{v_2^2}{2t^2}-\frac{u^{\alpha+2}}{\alpha+2},T=\frac{uv_2-t^2}{t}.$

(b) $F(u)=(\alpha+1)u^{\alpha}$	$\mathbf{UV}_1$	$\alpha\neq-1$ $\alpha\neq-2$	$\Lambda_1=e^{x}u^{\alpha+1},\Lambda_2=e^{x}v_1,$	$X=-e^{x}u^{\alpha+1}v_1,$ $T=e^{x}\left(\frac{u^{\alpha+2}}{\alpha+2}+\frac{v_1^2}{2}\right).$
$G(u)=u^{\alpha+1}$	$\mathbf{UV}_2$	$\alpha=-4$	$\Lambda_1=-e^{x}\frac{t}{u^3},\Lambda_2=e^{x}v_2.$	$X=e^{x}\frac{tv_2}{u^3},T=e^{x}\left(\frac{t^2}{u^2}-v_2^2\right).$
	$\mathbf{UB}_3$	$\alpha\neq-1$	$\Lambda_1=-u^{\alpha+1},\Lambda_2=e^{-x}b_3.$	$X=-u^{\alpha+1}b_3,T=e^{x}\frac{u^{\alpha+2}}{\alpha+2}+e^{-x}\frac{b_3^2}{2}.$
	$\mathbf{UB}_4$	$\alpha=-4$	$\Lambda_1=-\frac{t}{u^3},\Lambda_2=e^{-x}b_4.$	$X=-\frac{tb_4}{u^3},T=\frac{1}{2}e^{-x}b_4^2-e^{x}\frac{t^2}{2u^2}.$
(c) $F(u)=u^{\alpha}$ $G(u)=u$	$\mathbf{UV}_1$	$\alpha=1$	$\Lambda_1=\frac{t^4}{12}-xt^2+tv_1-\frac{u^2}{2}+x^2,$ $\Lambda_2=-\frac{t^3}{3}+t(u+2x)-v_1.$	$X=\left(\frac{v_1}{2}-xt+\frac{t^3-2tu}{6}\right)u^2$ $-\left(tv_1+\frac{t^4}{12}-xt^2+x^2\right)v_1,$ $T=-\frac{u^3}{6}+\left(\frac{t^4}{12}+x^2-xt^2+tv_1\right)u$ $+\left(2xt-\frac{v}{2}-\frac{t^3}{3}\right)v_1.$
			$\Lambda_1=\frac{t^3}{6}-xt+v_1$ $\Lambda_2=-\frac{t^2}{2}+u+x.$	$X=\left(\frac{t^2}{2}-\frac{u}{3}-x\right)u^2+\left(2xt-\frac{t^3}{3}-\frac{v_1}{2}\right)v_1,$ $T=\left(\frac{t^3}{3}-2xt\right)u+(u+2x-t^2)v_1.$
	$\mathbf{UV}_2$	$\alpha=1$	$\Lambda_1=\frac{t^2}{4}-x+\frac{v_2-x^2}{t^2},$ $\Lambda_2=t-\frac{u+2x}{t}.$	$X=\frac{u^3}{3}+\frac{2x-t^2}{2}u^2+\frac{v_2^2}{2t^2}+\frac{(t^4-4x(t^2+x))v_2}{4t^2},$ $T=-\frac{uv_2}{t}-\frac{(t^4-4x(t^2+x))u}{4t}-\frac{(2x-t^2)v_2}{t}.$
	$\mathbf{UC}_3$	$\alpha=1$	$\Lambda_1=-\frac{t^2-2x}{80}+\frac{2xt^2+5u^2}{40(t^2-2x)}+\frac{4x^3+5tc_3}{10(t^2-2x)^2},$ $\Lambda_2=\frac{3t^5-20c_3}{40(t^2-2x)^2}-\frac{t(2x+u)}{4(t^2-2x)}.$	$X=-\frac{(t^2-2x)(tu^2+2c_3)}{64}+\frac{t(u^3+3tc_3)}{48}+\frac{t^4(tu^2-10c_3)+20u^2c_3}{160(t^2-2x)}+\frac{t(t^5+5c_3)c_3}{40(t^2-2x)^2},$ $T=\frac{(t^4-4x^2)u}{64}+\frac{u^3-3t^4u-6tc_3}{96}+\frac{t(t^5+10c_3)u}{80(t^2-2x)}+\frac{(t^5+5c_3)c_3}{40(t^2-2x)^2}.$

Consider now a classification problem for the nonlinear telegraph (NLT) equation

$$u_{tt} - (F(u)u_x)_x - (G(u))_x = 0. \qquad (1)$$

For **any** $(F(u), G(u))$ pair, we naturally obtain potential systems

$$\begin{aligned} R_1[u,v] &= v_t - F(u)u_x - G(u) = 0, \\ R_2[u,v] &= v_x - u_t = 0; \end{aligned} \qquad (2)$$

$$\begin{aligned} H_1[u,v,w] &= R_1[u,v] = 0, \\ H_2[u,v,w] &= w_t - v = 0, \\ H_3[u,v,w] &= w_x - u = 0. \end{aligned} \qquad (3)$$

For **specific** $(F(u), G(u))$ pairs, the CL classification problem for (2), etc. can yield additional CLs and hence further potential systems for consideration [B & Temuerchaolu, J. Math. Anal. Appl. **310**, 459 (2005)]

NONLOCAL SYMMETRIES

$$X = \xi(x,t,U,V)\frac{\partial}{\partial x} + \tau(x,t,U,V)\frac{\partial}{\partial t} + \eta(x,t,U,V)\frac{\partial}{\partial U} + \phi(x,t,U,V)\frac{\partial}{\partial V}$$

is a point symmetry of the potential system (2) if and only if

$$\xi_V - \tau_U = 0,$$
$$\eta_U - \phi_V + \xi_x - \tau_t = 0,$$
$$G(U)[\eta_V + \tau_x] + \eta_t - \phi_x = 0,$$
$$\xi_U - F(U)\tau_V = 0,$$
$$\phi_U - G(U)\tau_U - F(U)\eta_V = 0,$$
$$G(U)\xi_V + \xi_t - F(U)\tau_x = 0,$$
$$F(U)[\phi_V - \tau_t + \xi_x - \eta_U - 2G(U)\tau_V] - F'(U)\eta = 0,$$
$$G(U)[\phi_V - \tau_t - G(U)\tau_V] - F(U)\eta_x - G'(U)\eta + \phi_t = 0,$$

holds for **arbitrary** values of x,t,U,V

Theorem 1 [B, Temuerchaolu & Sahadevan, JMP **46**, 023505 (2005)] The potential system (2) yields a nonlocal symmetry of the NLT eqn (1) if and only if

$$(c_3 u + c_4)F'(u) - 2(c_1 - c_2 - G(u))F(u) = 0,$$
$$(c_3 u + c_4)G'(u) + G^2(u) - (c_1 - 2c_2 + c_3)G(u) - c_5 = 0.$$

In the linearizable case: $c_1 = 0,\ c_5 = c_2(c_3 - c_2)$.

For any such pair ($F(u)$, $G(u)$), the (u,v) potential system has the point symmetry

$$X = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + \phi \frac{\partial}{\partial v}$$

with

$$\xi = c_1 x + \int F(u)du,$$
$$\tau = c_2 t + v,$$
$$\eta = c_3 u + c_4,$$
$$\phi = c_5 t + (c_1 - c_2 + c_3)v$$

Modulo translations and scalings in u and G and scalings in F (involving 5/7 parameters), one obtains six distinct classes of ODEs for $(F(u), G(u))$ where the scalar (u) eqn (1) has a potential symmetry.

Classification Table for Potential Symmetries

relationship	$G(u)$	$F(u)$
$F(u)=\dfrac{u^{\beta}}{\alpha}G'(u)$	$\dfrac{u^{2\alpha}-1}{u^{2\alpha}+1}$	$\dfrac{4u^{2\alpha+\beta-1}}{(u^{2\alpha}+1)^2}$
	$\dfrac{u^{2\alpha}+1}{u^{2\alpha}-1}$	$-\dfrac{4u^{2\alpha+\beta-1}}{(u^{2\alpha}-1)^2}$
$F(u)=\dfrac{u^{\beta}}{\alpha}G'(u)$	$\tan(\alpha\ln u)$	$u^{\beta-1}\sec^2(\alpha\ln u)$
$F(u)=u^{\beta}G'(u)$	$(\ln u)^{-1}$	$-u^{\beta-1}(\ln u)^{-2}$
$F(u)=e^{2\beta u}G'(u)$	$\tan u$	$e^{2\beta u}\sec^2 u$
$F(u)=e^{2\beta u}G'(u)$	$\tanh u$	$e^{2\beta u}\operatorname{sech}^2 u$
	$\coth u$	$-e^{2\beta u}\operatorname{csch}^2 u$
$F(u)=e^{2\beta u}G'(u)$	u^{-1}	$-u^{-2}e^{2\beta u}$

Modulo scalings and translations, two distinct linearization cases occur:

Case 1.

$$v_t - F(u)u_x = 0,$$
$$v_x - u_t = 0$$

admits

$$X = A(u,v)\frac{\partial}{\partial x} + B(u,v)\frac{\partial}{\partial t}$$

with

$$A_u - F(u)B_v = 0,$$
$$A_v - B_u = 0.$$

(hodograph transf)

Case 2.

$$v_t - u^{-2}u_x - u^{-1} = 0, v_x - u_t = 0$$

admits

$$X = -u^{-1}A(\hat{u},v)\frac{\partial}{\partial x} + B(\hat{u},v)\frac{\partial}{\partial t} + A(\hat{u},v)\frac{\partial}{\partial u},$$

$$\hat{u} = x + \log u,$$

with

$$A_v + B_{\hat{u}} = 0,$$

$$A_{\hat{u}} + B_v - A = 0.$$

Point Symmetry Classification of (*u*) Scalar NLT

Consider (*u*) scalar NLT eqn

$$u_{tt} = [F(u)u_x]_x + [G(u)]_x$$

Then

$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2),$$

$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2),$$

$$u^* = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2),$$

is a point symmetry of the (*u*) scalar eqn iff

$$X^{(2)}(u_{tt} - (F(u)u_x)_x - G(u)_x) = 0$$

for *any* soln of the (*u*) scalar eqn where $X^{(2)}$ is second extension of

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}$$

This leads to **determining equations**

$$\begin{aligned}
&\xi_u = \tau_x = \tau_u = \eta_{uu} = \xi_t = 0,\\
&2F(u)[-\tau_t + \xi_x] - F'(u)\eta = 0,\\
&\eta_{tt} - F(u)\eta_{xx} - G'(u)\eta_x = 0,\\
&2\eta_{tu} - \tau_{tt} = 0,\\
&F(u)[2\eta_{xu} - \xi_{xx}] + \xi_{tt} + 2F'(u)\eta_x - G'(u)[\xi_x - 2\tau_t] + G''(u)\eta = 0,
\end{aligned}$$

which must hold for *arbitrary* values of x, t, and u.

For arbitrary ($F(u)$, $G(u)$), translations in x and t are symmetries.

Classes of $(F(u), G(u))$ yielding point symmetries of scalar (*u*) NLT eqn

$G(u)$	$F(u)$	Infinitesimals
e^u	$e^{(\alpha+1)u}$	$(\xi,\tau,\eta)=(2\alpha x,[\alpha-1]t,2)$
$u^{\alpha+\beta+1}$	u^{α}	$(\xi,\tau,\eta)=(2\beta x,[\alpha+2\beta]t,-2u)$
u^{-1}	u^{-2}	above $+(\xi,\tau,\eta)=(e^x,0,-ue^x)$
$\ln u$	u^{α}	$(\xi,\tau,\eta)=(2[\alpha+1]x,[\alpha+2]t,2u)$
u	$e^{\alpha u}$	$(\xi,\tau,\eta)=(2\alpha x,\alpha t,2)$
u^{-3}	u^{-4}	above $+(\xi,\tau,\eta)=(0,t^2,tu)$

Theorem 2 Each point symmetry of the (u,v) NLT potential system that is a nonlocal symmetry of the NLT scalar (u) eqn yields a contact symmetry of the NLT (w) potential eqn given by

$$w_{tt} = F(w_x)w_{xx} + G(w_x)$$

Theorem 3 A point symmetry of the NLT scalar (u) eqn yields a point symmetry of the (u,v) NLT potential system for all cases except when $(F(u),G(u)) = (u^{-4},u^{-3})$.

CONSERVATION LAWS

$(\xi(x,t,U,V),\phi(x,t,U,V))$ are multipliers for a CL of NLT potential system iff

$$E_U\left(\xi R_1[U,V]+\phi R_2[U,V]\right)\equiv 0,$$
$$E_V\left(\xi R_1[U,V]+\phi R_2[U,V]\right)\equiv 0,$$

for **arbitrary** diff. functions $(U(x,t),(V(x,t))$. This yields determining eqns:

$$\begin{aligned}
&\phi_V-\xi_U=0,\\
&\phi_U-F(U)\xi_V=0,\\
&\phi_x-\xi_t-G(U)\xi_V=0,\\
&F(U)\xi_x-\phi_t-[G(U)\xi]_U=0.
\end{aligned}\qquad (4)$$

Then for any solution of (4), the conserved densities are

$$X = -\int_a^U \xi(x,t,s,b)ds - \int_b^V \phi(x,t,U,s)ds - G(a)\int^x \xi(s,t,a,b)ds,$$

$$T = \int_a^U \phi(x,t,s,b)ds + \int_b^V \xi(x,t,U,s)ds.$$

Classification results for CLs

Solution of determining system reduces to study of system of two functions

$$d(U) = G'^2 F''' - 3G'G''F'' + [3G''^2 - G'G''']F',$$

$$h(U) = G'^2 G^{(4)} - 4G'G''G''' + 3G''^3$$

Three cases arise:

$$d(U) = h(U) \equiv 0,$$

$$d(U) \not\equiv 0, h(U) \equiv 0,$$

$$d(U) \not\equiv 0, h(U) \not\equiv 0.$$

Case I: $F(u)$ is arbitrary

$F(u)$	$G(u)$	Multipliers
arb	u	$(\xi,\phi)=\left(t,x-\frac{1}{2}t^2\right)$ $(\xi,\phi)=\left(1,-t\right)$
arb	$1/u$	$(\xi,\phi)=\left(U,V\right)$ $(\xi,\phi)=(UV,\frac{1}{2}V^2+x+\int^U sF(s)ds$

Case II: $h(u) \neq 0, d(u) \neq 0$

$\gamma F - G'$ $= \frac{\alpha}{\gamma}(G+\beta)^2$	$(\xi_1, \phi_1) = Ae^{\sqrt{\alpha}(\beta t+V)}(1, \frac{\sqrt{\alpha}}{\gamma}(G+\beta))$ $(\xi_2, \phi_2) = (\xi_1, -\phi_1)(x, -t, U, -V)$ $[A = \exp(\gamma x + \frac{\alpha}{\gamma}\int^{U}(G(s)+\beta)ds]$
$\gamma F - G' = \frac{\alpha}{\gamma}$	$(\xi_1, \phi_1) = e^{\gamma x+\sqrt{\alpha}t}(1, \frac{\sqrt{\alpha}}{\gamma})$ $(\xi_2, \phi_2) = (\xi_1(x,-t), -\phi_1(x,-t))$
$\gamma F = G'$	$(\xi, \phi) = e^{\gamma x}(t, \frac{1}{\gamma})$ $(\xi, \phi) = e^{\gamma x}(V, \frac{1}{\gamma}G(U))$ $(\xi, \phi) = e^{\gamma x}(1,0)$

Case III: $d(u) = 0, h(u) = 0$

Using symmetry analysis (substitution + invariance under solvable three-parameter group), ODE $h(u) = 0$
can be solved in terms of elementary functions (for $G(u)$).

Then note that $F(u) = G(u) + \text{const}$ is a particular soln of resulting linear ODE $d(u) = 0 \Rightarrow$ general soln.

Consequently, for $F(u) = \beta_1 G^2(u) + \beta_2 G(u) + \beta_3,\ \beta_2{}^2 \neq 4\beta_1\beta_3$

there are four highly nontrivial conservation laws when

$$G(u) = u, \quad 1/u, \quad e^u, \quad \tanh u, \quad \tan u.$$

[In the case of a "perfect square" $\beta_2{}^2 = 4\beta_1\beta_3$, there are two conservation laws.]

3. Planar Gas Dynamics (PGD) Equations

Suppose the given system of PDEs is the planar gas dynamics (PGD) equations. In the *Eulerian description*, one has the **Euler system**

$$\mathbf{E}\{x,t,v,p,\rho\}=0: \quad \begin{cases} \rho_t+(\rho v)_x=0, \\ \rho(v_t+vv_x)+p_x=0, \\ \rho(p_t+vp_x)+B(p,\rho^{-1})v_x=0; \end{cases}$$

in terms of entropy density $S(p,\rho)$, constitutive function $B(p,\rho^{-1})$ is given by

$$B(p,\rho^{-1})=-\rho^2 S_\rho / S_p$$

In the *Lagrangian description*, in terms of the Lagrange mass coordinates $s = t,\ y = \int_{x_0}^{x} \rho(\xi) d\xi,$ one has the **Lagrange system**

$$\mathbf{L}\{y, s, v, p, q\} = 0: \quad \begin{cases} q_s - v_y = 0, \\ v_s + p_y = 0, \\ p_s + B(p, q) v_y = 0 \end{cases}$$

We now show that the potential system framework yields a direct connection between the Euler and Lagrange systems. As well, we derive other equivalent descriptions!

We use the Euler system as the given system. The first equation is a CL and through it, we introduce a potential variable r and obtain the potential system

$$\mathbf{G}\{x,t,v,p,\rho,r\}=0: \quad \begin{cases} r_x-\rho=0, \\ r_t+\rho v=0, \\ \rho(v_t+vv_x)+p_x=0, \\ \rho(p_t+vp_x)+B(p,\rho^{-1})v_x=0 \end{cases}$$

In order to obtain a nonlocally related subsystem, first consider an interchange of dependent and independent variables in **G** with $r = y$ and $t = s$ as independent variables; x, v, p, ρ as dependent variables and let $q = 1/\rho$ to obtain the 1:1 equivalent system

$$\mathbf{G}_0\{y,s,x,v,p,\rho\}=0: \quad \begin{cases} x_y-q=0, \\ x_s-v=0, \\ v_s+p_y=0, \\ p_s+B(p,q)v_y=0 \end{cases}$$

A nonlocally related subsystem of $\mathbf{G_0}$ is obtained by excluding x through
$x_{ys} = x_{sy}$ to obtain the **Lagrange system** $\mathbf{L}\{y, s, v, p, q\} = 0$

A second conservation law of the Euler system obtained with the multipliers

$$(\Lambda_1, \Lambda_2, \Lambda_3) = (v, 1, 0)$$

yields a second potential variable w. The couplet potential system containing both of the obtained variables r and w is given by

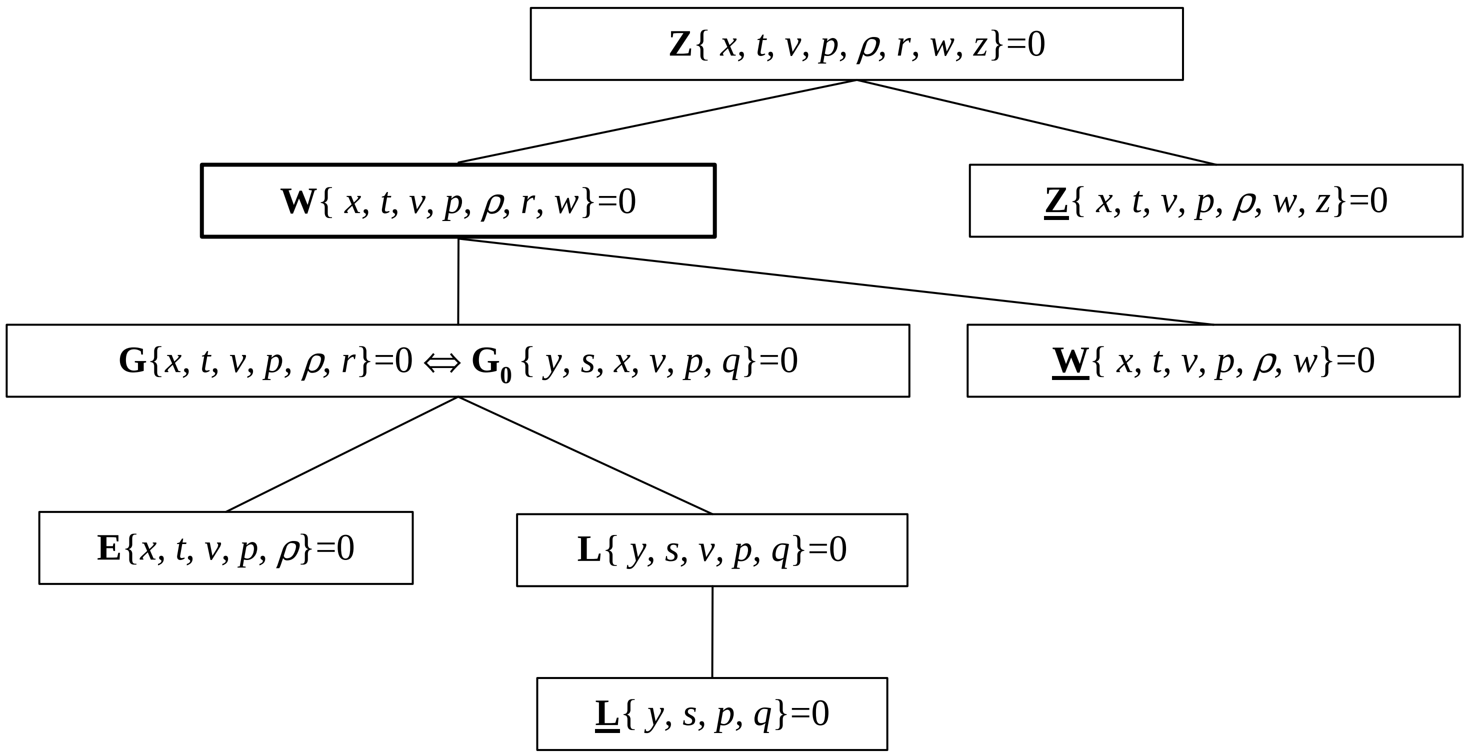
$$\mathbf{W}\{x, t, v, p, \rho, r, w\} = 0: \quad \begin{cases} r_x - \rho = 0, \\ r_t + \rho v = 0, \\ w_x + r_t = 0, \\ w_t + p + v w_x = 0, \\ \rho(p_t + v p_x) + B(p, \rho^{-1}) v_x = 0 \end{cases}$$

From the third equation of **W**, one can introduce a third potential variable z
and obtain potential system

$$\mathbf{Z}\{x,t,v,p,\rho,r,w,z\}=0: \quad \begin{cases} r_x-\rho=0, \\ r_t+\rho v=0, \\ z_t-w=0, \\ z_x+r=0, \\ w_t+p+vw_x=0, \\ \rho(p_t+vp_x)+B(p,\rho^{-1})v_x=0 \end{cases}$$

The Lagrangian system **L**, has a nonlocally related subsystem

$$\underline{\mathbf{L}}\{y,s,p,q\}=0: \quad \begin{cases} q_{ss}+p_{yy}=0, \\ p_s+B(p,q)q_s=0 \end{cases}$$



Tree of nonlocally related systems for PGD equations

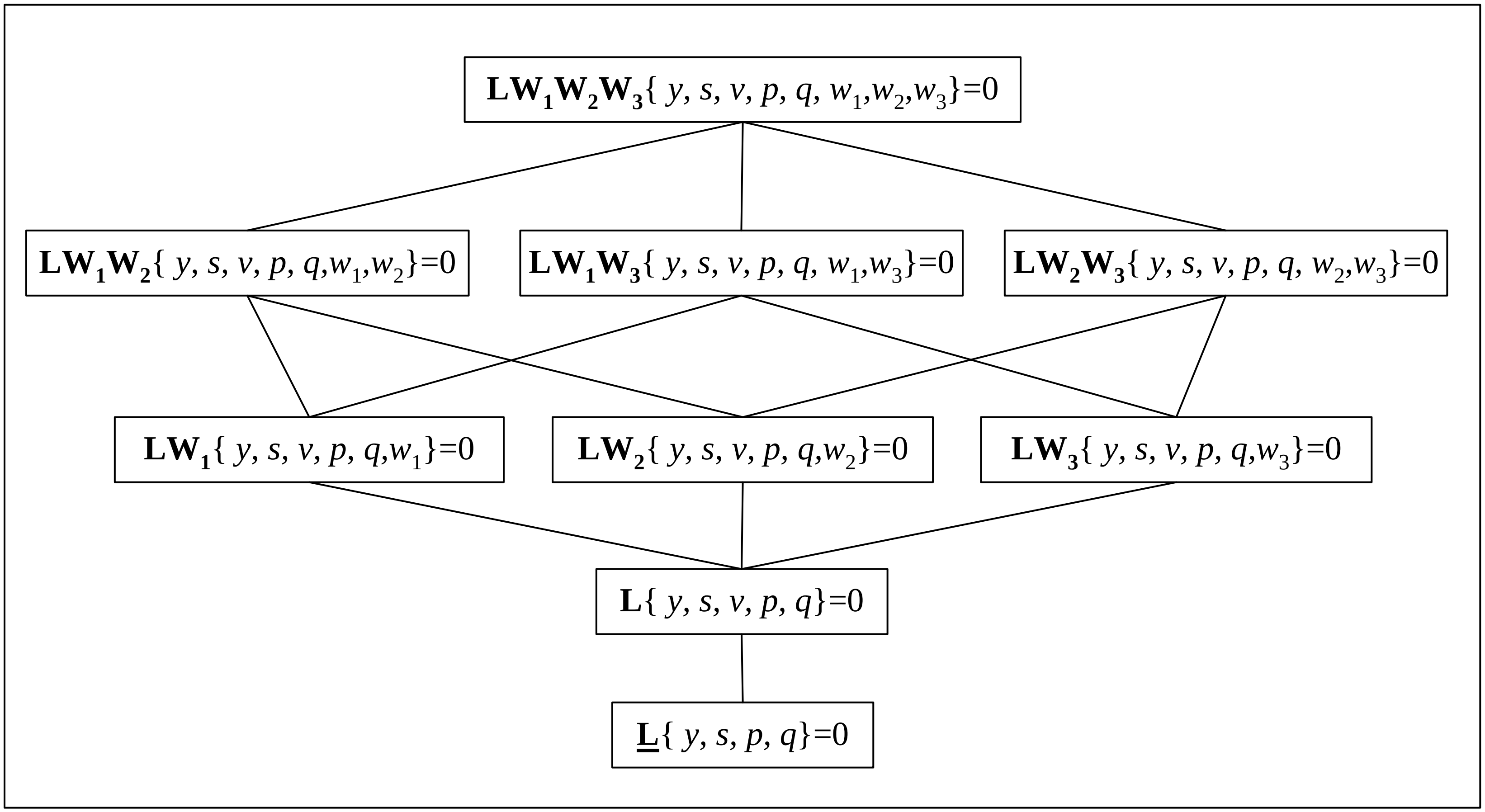
Starting from the Lagrange system $\mathbf{L}$, one can obtain three singlet potential systems from three sets of multipliers

$$(\mu_1(y,s),\mu_2(y,s),\mu_3(y,s))=(1,0,0),(0,1,0),(y,s,0)$$

$$\mathbf{G}=0 \Leftrightarrow \mathbf{G}_0=\mathbf{LW}_1\{y,s,v,p,q,w_1\}=0:\begin{cases} w_{1_y}-q=0, \\ w_{1_s}-v=0, \\ v_s+p_y=0, \\ p_s+B(p,q)v_y=0 \end{cases}$$

$$\mathbf{LW}_2\{y,s,v,p,q,w_2\}=0:\begin{cases} q_s-v_y=0, \\ w_{2_y}-v=0, \\ w_{2_s}+p=0, \\ p_s+B(p,q)v_y=0 \end{cases}$$

$$\mathbf{LW}_3\{y,s,v,p,q,w_3\}=0:\begin{cases} w_{3_y}-sv-yq=0, \\ w_{3_s}+sp-yv=0, \\ v_s+p_y=0, \\ p_s+B(p,q)v_y=0 \end{cases}$$



Extension of tree of nonlocally related systems for Lagrange system for PGD eqns

Two more CLs arise for the Lagrange system **L**, when one considers multipliers of the form

$$\mu_i(y, s, V, P, Q),\ i = 1,2,3.$$

In general, one can show that

$$\begin{aligned}
\mu_1 &= \alpha y - \beta P + B(P,Q)\mu_3 + \delta,\\
\mu_2 &= \alpha s + \beta V + \nu,\\
\mu_3 &= \mu_3(y, P, Q),
\end{aligned}$$

where $\alpha, \beta, \nu, \delta$ are abitrary constants and $\mu_3(y,P,Q)$ is any solution of PDE

$$\frac{\partial \mu_3}{\partial Q} - \frac{\partial}{\partial P}(B(P,Q)\mu_3) + \beta = 0$$

The two extra CLs arise (for *arbitrary* constitutive function $B(p,q)$)

(1) from conservation of energy

$$\frac{\partial}{\partial s}(\tfrac{1}{2}v^2 + K(p,q)) + \frac{\partial}{\partial y}(pv) = 0$$

where $K(p,q)$ is a solution of eqn

$$K_q - B(p,q)K_p + p = 0$$

(2) from conservation of entropy

$$\frac{\partial}{\partial s}S(p,q) = 0$$

where $S(p,q)$ is a solution of eqn

$$S_q - B(p,q)S_p = 0$$

For multipliers restricted to dependence on the independent variables (y,s), no further potential systems (just the first three) arise in the case of a Lagrange PGD system **L** with a generalized polytropic equation of state

$$B(p,q)=\frac{M(p)}{q}, \quad M''(p)\neq 0$$

TABLE VII. Symmetries of the generalized polytropic PGD system (2.10), (5.1).

System	$M(p)$	Symmetries
L, **LW**$_1$,**LW**$_2$,**LW**$_3$, **LW**$_1$**W**$_2$,**LW**$_1$**W**$_3$, **LW**$_2$**W**$_3$, **LW**$_1$**W**$_2$**W**$_3$	(i) Arbitrary	$Z_1=\frac{\partial}{\partial s}+w_2\frac{\partial}{\partial w_3}$, $Z_2=\frac{\partial}{\partial y}+w_1\frac{\partial}{\partial w_3}$, $Z_3=\frac{\partial}{\partial v}+s\frac{\partial}{\partial w_1}+y\frac{\partial}{\partial w_2}+sy\frac{\partial}{\partial w_3}$, $Z_4=-y\frac{\partial}{\partial y}+2q\frac{\partial}{\partial q}+v\frac{\partial}{\partial v}+w_1\frac{\partial}{\partial w_1}$, $Z_5=s\frac{\partial}{\partial s}+y\frac{\partial}{\partial y}+w_1\frac{\partial}{\partial w_1}+w_2\frac{\partial}{\partial w_2}+2w_3\frac{\partial}{\partial w_3}$, $Z_6=\frac{\partial}{\partial w_1}$, $Z_7=\frac{\partial}{\partial w_2}$, $Z_8=\frac{\partial}{\partial w_3}$.
L,**LW**$_2$	(ii) $-p\ln p$	$Z_9=y\frac{\partial}{\partial y}+2p\frac{\partial}{\partial p}+\frac{2q}{\ln p}\frac{\partial}{\partial q}+v\frac{\partial}{\partial v}+2w_2\frac{\partial}{\partial w_2}$.
	(iii) $\gamma p+\alpha p^{(\gamma+1)/\gamma}$ $\gamma\neq 0,-1$	$Z_{10}=\frac{(\gamma+1)y}{2\gamma}\frac{\partial}{\partial y}+p\frac{\partial}{\partial p}-\frac{q}{\delta p^{1/\gamma}+\gamma}\frac{\partial}{\partial q}+\frac{(\gamma-1)v}{2\gamma}\frac{\partial}{\partial v}+w_2\frac{\partial}{\partial w_2}$.
	(iv) $1+\alpha e^p$, $\alpha=\pm 1$	$Z_{11}=\frac{\partial}{\partial p}+\frac{\alpha e^p}{1+\alpha e^p}q\frac{\partial}{\partial q}-s\frac{\partial}{\partial w_2}$, $Z_{12}=y\frac{\partial}{\partial p}+\frac{\alpha e^p}{1+\alpha e^p}yq\frac{\partial}{\partial q}-s\frac{\partial}{\partial v}-sy\frac{\partial}{\partial w_2}$.
LW$_2$	(ii) $-p\ln p$	$Z_{13}=y^2\frac{\partial}{\partial y}+yp\frac{\partial}{\partial p}-\left(3-\frac{1}{\ln p}\right)yq\frac{\partial}{\partial q}-(yv-w_2)\frac{\partial}{\partial v}+yw_2\frac{\partial}{\partial w_2}$.
	(iii) $\gamma p+\delta p^{(\gamma+1)/\gamma}$ $\gamma\neq 0,-1$	$Z_{14}=y^2\frac{\partial}{\partial y}+yp\frac{\partial}{\partial p}-\left(3-\frac{\delta\ \ p^{1/\gamma}}{\gamma\,\delta p^{1/\gamma}+\gamma}\right)yq\frac{\partial}{\partial q}-(yu-w_2)\frac{\partial}{\partial v}+yw_2\frac{\partial}{\partial w_2}$.

TABLE VIII. Point symmetries of the subsystem $\underline{\mathrm{L}}$ (2.19) of the generalized polytropic PGD system (2.10), (5.1).

$M(p)$	Symmetries
(i) Arbitrary	$\underline{Z_1}=\frac{\partial}{\partial s},\ \underline{Z_2}=\frac{\partial}{\partial y}$, $\underline{Z_4}=-y\frac{\partial}{\partial y}+2q\frac{\partial}{\partial q},\ \underline{Z_5}=s\frac{\partial}{\partial s}+y\frac{\partial}{\partial y}$.
(ii) $-p\ln p$	$\underline{Z_9}=y\frac{\partial}{\partial y}+2p\frac{\partial}{\partial p}+\frac{2q}{\ln p}\frac{\partial}{\partial q}$, $\underline{Z_{13}}=y^2\frac{\partial}{\partial y}+yp\frac{\partial}{\partial p}-\left(3-\frac{1}{\ln p}\right)yq\frac{\partial}{\partial q}$.
(iii) $\gamma p+\delta p^{(\gamma+1)/\gamma}$	$\underline{Z_{10}}=\frac{(\gamma+1)y}{2\gamma}\frac{\partial}{\partial y}+p\frac{\partial}{\partial p}-\frac{q}{\delta p^{1/\gamma}+\gamma}\frac{\partial}{\partial q}$, $\underline{Z_{14}}=y^2\frac{\partial}{\partial y}+yp\frac{\partial}{\partial p}-\left(3-\frac{\alpha}{\gamma}\frac{p^{1/\gamma}}{\delta p^{1/\gamma}+\gamma}\right)yq\frac{\partial}{\partial q}$.
$\gamma=3$	$\underline{Z_{15}}=\frac{1}{3}s^2\frac{\partial}{\partial s}-sp\frac{\partial}{\partial p}+\frac{1}{\delta p^{4/3}+3}spq\frac{\partial}{\partial q}$.
(iv) $1+\alpha e^p$	$\underline{Z_{11}}=\frac{\partial}{\partial p}+\frac{\alpha e^p}{1+\alpha e^p}q\frac{\partial}{\partial q}$, $\underline{Z_{12}}=y\frac{\partial}{\partial p}+\frac{\alpha e^p}{1+\alpha e^p}yq\frac{\partial}{\partial q}$.

Remarks

- Extended trees hold for an *arbitrary* constitutive function

- Either Euler system or Lagrange system can be given system—tree will not change

- In Akhatov, Gazizov & Ibragimov (1991) a complete group classification with respect to constitutive function was given separately for Euler and Lagrange systems but connections between systems were heuristic

- To systematically construct nonlocal symmetries of the Euler and Lagrange systems one needs to do the group classification problem for *all* systems in an extended tree with respect to the constitutive function (as well as consider other possible extended trees for specific constitutive functions followed by point symmetry analysis)

- E.g., for Chaplygin gas $[B(p,q] = -p/q]$, subsystem

$$\underline{\underline{\mathbf{L}}}\{y, s, p, q\} = 0$$

has the point symmetry (not in AGI) given by

$$\mathrm{X} = -y^2 \frac{\partial}{\partial y} - py \frac{\partial}{\partial p} + 3yq \frac{\partial}{\partial q},$$

which yields a nonlocal symmetry for **E** and **L.**

Further extended trees for PGD eqns for specific constitutive functions

Example A: For $B(p,1/\rho)=\rho(1+e^{p})$, system $\mathbf{G}\{x,t,v,p,\rho,r\}=0$ has a family of CLs:

$$D_t\left(\frac{f(r)e^{p}}{1+e^{p}}\right)+D_x\left(\frac{f(r)ve^{p}}{1+e^{p}}\right)=0,$$

for arbitrary $f(r)$. This CL can be used to replace the 4th eqn of $\mathbf{G}\{x,t,v,p,\rho,r\}=0$ to introduce potential variable c and potential system

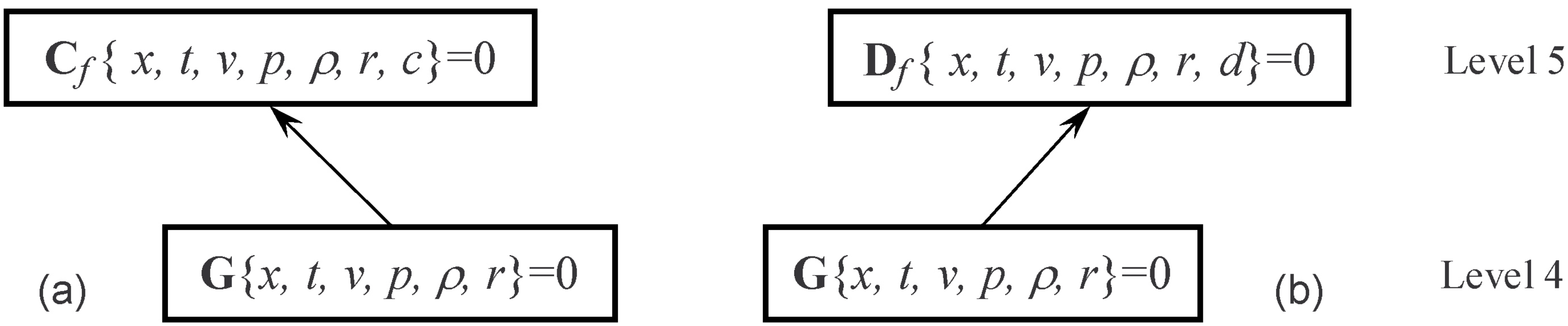
$$\mathbf{C}_f\{x,t,v,p,\rho,r,c\}=0:\begin{cases} r_x-\rho=0,\\ r_t+\rho v=0,\\ r_x(v_t+vv_x)+p_x=0,\\ c_x+e^{p}f(r)/(1+e^{p})=0,\\ c_t-ve^{p}f(r)/(1+e^{p})=0.\end{cases}$$

Example B: For Chaplygin gas, $B(p,1/\rho) = -p\rho$, system $\mathbf{G}\{x,t,v,p,\rho,r\} = 0$ has a family of CLs**:**

$$D_t\left(\frac{f(r)}{p}\right) + D_x\left(\frac{f(r)v}{p}\right) = 0$$

for arbitrary $f(r)$ to yield a family of potential systems

$$\mathbf{D}_f\{x,t,v,p,\rho,r,d\} = 0: \begin{cases} r_x - \rho = 0, \\ r_t + \rho v = 0, \\ r_x(v_t + vv_x) + p_x = 0, \\ d_x + f(r)/p = 0, \\ d_t - vf(r)/p = 0. \end{cases}$$



$\mathbf{C}_f\{x, t, v, p, \rho, r, c\}=0$
$\mathbf{D}_f\{x, t, v, p, \rho, r, d\}=0$
Level 5
(a)
$\mathbf{G}\{x, t, v, p, \rho, r\}=0$
$\mathbf{G}\{x, t, v, p, \rho, r\}=0$
(b)
Level 4

One can show that new nonlocal symmetries arise for the Chaplygin gas Euler system only when $f(r)=r,\ f(r)=\text{const.}$
For $f(r)=r,$ this Chaplygin gas system has symmetries

$$\mathrm{X}_{\mathbf{D}_1} = \left(-\frac{t^3}{6}+dt\right)\frac{\partial}{\partial x}+\left(d-\frac{t^2}{2}\right)\frac{\partial}{\partial v}+rt\frac{\partial}{\partial p}-\frac{rt\rho}{p}\frac{\partial}{\partial p},$$

$$\mathrm{X}_{\mathbf{D}_2} = \left(-\frac{t^2}{2}+d\right)\frac{\partial}{\partial x}+-t\frac{\partial}{\partial v}+r\frac{\partial}{\partial p}-\frac{r\rho}{p}\frac{\partial}{\partial p}.$$

- Symmetry $\mathrm{X}_{\mathbf{D}_1}$ is nonlocal for both the Euler and Lagrange systems
- Symmetry $\mathrm{X}_{\mathbf{D}_2}$ is nonlocal for the Euler system but local for the Lagrange system
- Hence in AGI, symmetry $\mathrm{X}_{\mathbf{D}_1}$ was not obtained

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