Nonclassical Method for finding solutions of PDEs

Consider PDE system $\mathbf{R}\{x;u\}$ of N PDEs of order k with n independent variables $x=(x^1,\ldots,x^n)$ and m dependent variables $u=(u^1,\ldots,u^m)$, given by

$$R^\sigma(x,u,\partial u,\ldots,\partial^k u)=0, \quad \sigma=1,\ldots,N, \tag{1}$$

that has point symmetry with infinitesimal generator

$$\mathrm{X}=\xi^i(x,u)\frac{\partial}{\partial x^i}+\eta^\mu(x,u)\frac{\partial}{\partial u^\mu}, \tag{2a}$$

or, equivalently, in evolutionary form, infinitesimal generator

$$\hat{\mathrm{X}} = (\eta^{\mu}(x,u) - \xi^{i}(x,u)u_{i}^{\mu})\frac{\partial}{\partial u^{\mu}}. \qquad \text{(2b)}$$

Let $\xi(x,u) = (\xi^{1}(x,u),\ldots,\xi^{n}(x,u))$ and assume that $\xi(x,u) \not\equiv 0$.

Definition $u = \Theta(x)$, with components $u^{\nu} = \Theta^{\nu}(x), \nu = 1,\ldots,m,$ is a resulting ***invariant solution*** of PDE system $\mathbf{R}\{x;u\}$ if and only if

(i) $u^{\nu} = \Theta^{\nu}(x)$ is an invariant surface of point symmetry (2) for each $\nu = 1,\ldots,m$;

(ii) $u = \Theta(x)$ is a solution of $\mathbf{R}\{x;u\}$.

Hence $u = \Theta(x)$ is a resulting invariant solution of PDE system $\mathbf{R}\{x;u\}$ if and only if $u = \Theta(x)$ satisfies

$$\text{(i)}\ \mathrm{X}(u^\nu - \Theta^\nu(x)) = 0 \text{ when } u = \Theta(x), \nu = 1,\ldots,m \qquad \text{(3a)}$$

$$\leftrightarrow \mathrm{X}(u^\nu - \Theta^\nu(x)\Big|_{u=\Theta(x)}, \nu = 1,\ldots,m \qquad \text{(3b)}$$

$$\leftrightarrow \eta^\nu(x,\Theta(x)) - \xi^i(x,\Theta(x))\frac{\partial\Theta^\nu(x)}{\partial x^i} = 0, \nu = 1,\ldots,m \qquad \text{(3c)}$$

$$\leftrightarrow \hat{\mathrm{X}}u^\nu\Big|_{u=\Theta(x)} = 0, \nu = 1,\ldots,m; \qquad \text{(3d)}$$

$$\text{(ii)}\ R^\sigma(x,u,\partial u,\ldots,\partial^k u) = 0 \text{ when } u = \Theta(x), \sigma = 1,\ldots,N \qquad \text{(4a)}$$

$$\leftrightarrow R^\sigma(x,\Theta(x),\partial\Theta(x),\ldots,\partial^k\Theta(x)) = 0, \sigma = 1,\ldots,N \qquad \text{(4b)}$$

$$\leftrightarrow R^\sigma(x,u,\partial u,\ldots,\partial^k u)\Big|_{u=\Theta(x)} = 0, \sigma = 1,\ldots,N. \qquad \text{(4c)}$$

Equations (3) and (4) define the ***classical method*** to obtain particular solutions of a PDE system $\mathbf{R}\{x;u\}$.

Classical method

In summary, $u=\Theta(x)$ is a solution (invariant solution) of PDE system $\mathbf{R}\{x;u\}$ obtained through the ***classical method*** [Lie (1881)] if and only if there exists a Lie group of point transformations with infinitesimal generator X given by (2a) [$\hat{\mathrm{X}}$ given by (2b)], with kth extension (prolongation) $\mathrm{X}^{(k)}$, such that

$$\mathrm{X}^{(k)} R^{\sigma}(x,u,\partial u,\ldots,\partial^k u)\Big|_{R^{\lambda}(x,u,\partial u,\ldots,\partial^k u)=0,\ \lambda=1,\ldots,N} = 0,\ \sigma=1,\ldots,N;$$

$$\hat{\mathrm{X}}u^{\nu}\Big|_{u=\Theta(x)} = 0,\ \nu=1,\ldots,m; \qquad (6)$$

$$R^{\sigma}(x,u,\partial u,\ldots,\partial^k u)\Big|_{u=\Theta(x)} = 0,\ \sigma=1,\ldots,N. \qquad (7)$$

Having found point symmetry with infinitesimal generator (2a) through solving the linear system of determining equations (5), one can proceed in two ways to solve the systems of equations (6) and (7) to find an invariant solution $u = \Theta(x)$.

(1) ***Invariant form method***

Here one first solves invariant surface conditions (6) by explicitly solving corresponding characteristic equations for $u = \Theta(x)$

$$\frac{dx^1}{\xi^1(x,u)} = \cdots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^m}{\xi^m(x,u)}. \qquad (8)$$

If $z^1(x,u),\ldots,z^{n-1}(x,u),h^1(x,u),\ldots,h^m(x,u)$, are $n+m-1$ functionally independent constants of integration that arise from solving characteristic system of ODEs (8) with Jacobian $\partial(h^1,\ldots,h^m)/\partial(u^1,\ldots,u^m)\neq 0$, then the general solution $u=\Theta(x)$ of invariant surface condition equations (6) is given implicitly by invariant form

$$h^{\nu}(x,u)=H^{\nu}(z^1(x,u),\ldots,z^{n-1}(x,u)), \qquad (9)$$

where H^{ν} is an arbitrary differentiable function of its arguments, $\nu=1,\ldots,m.$ Note that $z^1(x,u),\ldots,z^{n-1}(x,u),h^1(x,u),\ldots,h^m(x,u)$, are $n+m-1$ functionally independent invariants of the one-parameter Lie group of point transformations with infinitesimal generator (2a) -> $n+m-1$ canonical coordinates for the one-parameter Lie group of point transformations with infinitesimal generator (2a).

Let $z^n(x,u)$ be the $(n + m)$th canonical coordinate satisfying $\mathrm{X}z^n = 1$. If PDE system $\mathbf{R}\{x;u\}$ is transformed by the corresponding invertible point transformation into PDE system $\mathbf{S}\{z;h\}$ with independent variables $z = (z^1, \ldots, z^n)$ and dependent variables $h = (h^1, \ldots, h^m)$, then transformed PDE system $\mathbf{S}\{z;h\}$ has translation symmetry

$$(z^*)^i = z^i, \quad i = 1, \ldots, n-1,$$

$$(z^*)^n = z^n + \varepsilon,$$

$$(h^*)^\nu = h^\nu, \quad \nu = 1, \ldots, m.$$

Thus variable z^n does not appear explicitly in transformed PDE system $\mathbf{S}\{z;h\}$, and hence the transformed PDE system has ***particular*** solutions of the form (9) that yield implicitly, ***specific*** functions $u=\Theta(x)$ which are invariant solutions of PDE system $\mathbf{R}\{x;u\}$, i.e., PDE system $\mathbf{R}\{x;u\}$ has invariant solutions implicitly given by invariant form (9).

These invariant solutions are found by solving a reduced system of DEs with $n-1$ independent variables $z^1,\ldots,z^{n-1}$ and m dependent variables $h^1,\ldots,h^m$. $z^1,\ldots,z^{n-1}$ are commonly called ***similarity variables.***

The reduced system of DEs is found by substituting invariant form (9) into given PDE system $\mathbf{R}\{x;u\}$. Note that if $\partial\xi/\partial u\equiv 0$, then $z^i=z^i(x), i=1,\ldots,n-1$. When $\mathbf{R}\{x;u\}$ (5.1) has two independent variables, the reduced system is an ODE system with independent variable $z=z^1$.

(2) ***Direct substitution method***

This procedure is essential if one is unable to solve explicitly the invariant surface condition equations (6), i.e., if one is unable to obtain the general solution of characteristic ODE system (8). One can assume $\xi^n(x,u) \neq 0$. Then (6) can be written as

$$\frac{\partial u^\nu}{\partial x^n} = \frac{\eta^\nu(x,u)}{\xi^n(x,u)} - \sum_{i=1}^{n-1} \frac{\xi^i(x,u)}{\xi^n(x,u)} \frac{\partial u^\nu}{\partial x^i}, \quad \nu = 1,\ldots,m. \qquad (10)$$

From (10) and differential consequences, any term involving derivatives of components of u with respect to independent variable x^n can be expressed in terms of components of x and u as well as derivatives of components of u with respect to the independent variables $x^1,\ldots,x^{n-1}$.

After directly substituting (10) and its differential consequences for any partial derivative with respect to x^n appearing in $\mathbf{R}\{x;u\}$, one obtains a reduced DE system directly involving m dependent variables $u^1,\ldots,u^m$, $n-1$ independent variables $x^1,\ldots,x^{n-1}$, derivatives of $u^1,\ldots,u^m$ with respect to $x^1,\ldots,x^{n-1}$, and ***parameter*** x^n.

A solution $u=\Phi(x^1,\ldots,x^{n-1};x^n)$ of this reduced DE system yields an invariant solution $u=\Theta(x)$ of $\mathbf{R}\{x;u\}$ provided that the invariant surface condition equations (6) or, equivalently the given PDE system $\mathbf{R}\{x;u\}$ itself, are also satisfied.

In the case of two independent variables, the reduced system of DEs is an ODE system.

Here the constants of integration in the general solution of the reduced ODE system are arbitrary functions of the parameter x^n, and these arbitrary functions are then determined by substituting this general solution into either the invariant surface condition equations (6) or the given PDE system $\mathbf{R}\{x;u\}$.

For examples of invariant solutions of PDEs, see the books of Ovsiannikov [(1962), (1982)], Bluman & Cole (1974), Olver (1986), Bluman & Kumei (1989), Stephani (1989), Hydon (2000), Bluman & Anco (2002) and Cantwell (2002).

Nonclassical method

The nonclassical method (B 1967), generalizes and includes Lie's classical method ***for obtaining solutions of PDEs.***

Here one first seeks functions $\xi^i(x,u), \eta^\mu(x,u), i=1,\ldots,n, \mu=1,\ldots,m,$ so that (2a) is a "symmetry" ***("nonclassical symmetry")*** of ***augmented PDE system*** $\mathbf{A}\{x;u\}$ consisting of $\mathbf{R}\{x;u\}$, invariant surface equations

$$I^\nu(x,u,\partial u) = \eta^\nu(x,u) - \xi^i(x,u)\frac{\partial u^\nu}{\partial x^i} = 0, \quad \nu = 1,\ldots,m, \qquad (11)$$

and differential consequences of (11).

Consequently, one obtains an overdetermined set of ***nonlinear*** determining equations for unknown functions $\xi^i(x,u), \eta^\mu(x,u), i=1,\ldots,n, \mu=1,\ldots,m.$

It is straightforward to show that, for any set of $\xi^i(x,u), \eta^\mu(x,u), i=1,\ldots,n, \mu=1,\ldots,m,$ (2a) is a symmetry of invariant surface condition equations (11)

From this it follows that the nonclassical method includes Lie's classical method.

Resulting set of determining equations is nonlinear due to substitution of equations (11) (each written in solved form with respect to some derivative term) and their differential consequences into the symmetry determining equations (5) that now hold only for solutions of the augmented PDE system.

In the nonclassical method, the invariant surface condition equations (11) are essentially a set of constraint equations of a particular form.

In particular, the nonclassical method is equivalent to seeking all solutions of $\mathbf{R}\{x;u\}$ of form (11) for ***any*** set of $\xi^i(x,u), \eta^\mu(x,u), i=1,\ldots,n, \mu=1,\ldots,m.$

Set of determining equations satisfied by $\xi^i(x,u), \eta^\mu(x,u), i=1,\ldots,n, \mu=1,\ldots,m,$ are the compatibility conditions for existence of solutions of augmented PDE system $\mathbf{A}\{x;u\}$ that includes $\mathbf{R}\{x;u\}$ and constraint equations (11).

A "nonclassical symmetry" is *not a symmetry* of $\mathbf{R}\{x;u\}$ unless the infinitesimals yielding an infinitesimal generator (2a) yield a point symmetry of $\mathbf{R}\{x;u\}$.

Otherwise, a mapping resulting from such an infinitesimal generator maps *no* solution of $\mathbf{R}\{x;u\}$ into a different solution of $\mathbf{R}\{x;u\}$. It just maps the solution obtained by the nonclassical method into itself!

Strictly speaking, the nonclassical method is not a “symmetry” method but an extension of Lie’s symmetry method (“classical method”) with the purpose of finding specific solutions of PDEs.

The situation for a scalar PDE with two independent variables

Now consider situation of a scalar PDE with two independent variables.

Let $x^1 = x,\ x^2 = t,\ \xi^1 = \xi(x,t,u),\ \xi^2 = \tau(x,t,u).$

Then invariant surface condition equations (11) become invariant surface condition equation

$$\xi(x,t,u)u_x + \tau(x,t,u)u_t = \eta(x,t,u). \qquad (12)$$

For a specific set of $\xi(x,t,u), \tau(x,t,u), \eta(x,t,u)$, the general solution of invariant surface condition (12) can be represented in the form

$$z(x,t,u) = \text{const} = c_1, \quad (\textbf{\textit{similarity variable}}) \qquad (13a)$$

$$H(x,t,u) = \text{const} = c_2 = h(z). \qquad (13b)$$

After solving equation (13b) for u, one obtains ***ansatz***

$$u = \phi(x,t,h(z(x,t,u))) \qquad (14)$$

for solutions of the given scalar PDE.

If a specific set of $(\xi(x,t,u), \tau(x,t,u), \eta(x,t,u))$ is a set of infinitesimals for a point symmetry of the scalar PDE, then the dependence of ϕ on x, t, and $h(z)$ is explicit in ansatz (14); $h(z)$ is an arbitrary function of the similarity variable z.

Here, substitution of the ansatz (14) into the scalar PDE yields a reduced ODE of order at most k with independent variable z and dependent variable $h(z)$.

Each solution of this ODE yields an invariant solution, obtainable by the classical method, of the scalar PDE.

If $\xi_u = \tau_u \equiv 0$, then $z(x,t,u) \equiv z(x,t)$, and ansatz (14) reduces to

$$u = \phi(x,t,h(z(x,t))). \tag{15}$$

If $\xi_u = \tau_u = \eta_{uu} \equiv 0$, ansatz (14) further reduces to

$$u = A(x,t) + B(x,t)h(z(x,t)). \qquad (16)$$

In the ansatz (16), functions $A(x,t)$ and $B(x,t)$ are explicitly known for a specific set of functions $(\xi(x,t), \tau(x,t), \eta(x,t,u))$.

Suppose one obtains the sets of all infinitesimals $(\xi(x,t,u), \tau(x,t,u), \eta(x,t,u))$ of symmetries $\mathrm{X} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}$ of augmented system $\mathbf{A}\{x;u\}$ consisting of the given scalar PDE, the constraint invariant surface condition equation (12), and the differential consequences of (12).

Then it follows that the solutions $u = \Phi(x,t)$ of the scalar PDE, arising from the nonclassical method, include ***all*** solutions of the scalar PDE of the form $u = \phi(x,t,h(z(x,t,u)))$ where $h(z)$ satisfies a reduced ODE.

Hence the nonclassical solutions of the scalar PDE include all solutions of the PDE obtained by the ***direct method*** of Clarkson & Kruskal (1989) since the direct method aims to find all solutions of a scalar PDE that are of the ansatz (14) with the restriction that $z(x,t,u) \equiv z(x,t)$ [Nucci & Clarkson (1992)].

From the nature of the constraint invariant surface condition equation (12), without loss of generality, in using the nonclassical method, two simplifying cases need only be considered when solving the determining equations for $(\xi(x,t), \tau(x,t), \eta(x,t,u))$, namely, $\tau \equiv 1$; $\tau \equiv 0, \xi \equiv 1$.

This follows from the observations that if

(1) $\tau \neq 0$, the constraint equation (12) can be divided by τ, and hence set $\tau \equiv 1$, so there are only two independent infinitesimals

(2) $\tau \equiv 0, \xi \neq 0$, the constraint equation (12) can be divided by ξ, and hence set $\xi \equiv 1$, so here there is only one independent infinitesimal.

Note that for a given set of infinitesimals $(\xi(x,t),\tau(x,t),\eta(x,t,u))$ that satisfy the nonlinear determining equations, one can use either the invariant form or direct substitution method to find the resulting solutions of the scalar PDE.

Examples

(1) ***Heat equation***

The first PDE considered through the nonclassical method was the linear heat equation [B (1967), B & Cole (1969)]

$$u_t - u_{xx} = 0. \qquad (17)$$

Case 1. The classical method

The classical method determining equation (5) that yields the point symmetries $X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}$ of the linear heat equation (17) is given by

$$\tau_{uu} u_x{}^2 u_t + \xi_{uu} u_x{}^3 + 2\tau_u u_{xt} u_x + 2(\tau_{xu} + \xi_u) u_x u_t + (2\xi_{xu} - \eta_{uu}) u_x{}^2 + 2\tau_x u_{xt}$$
$$+ (\tau_{xx} - \tau_t + 2\xi_x) u_t + (\xi_{xx} - \xi_t - 2\eta_{xu}) u_x + (\eta_t - \eta_{xx}) = 0. \tag{18}$$

Equation (18) must hold for arbitrary values of $x, t, u, u_t, u_x, u_{xt}$.

Thus (18) splits into nine equations.

This yields the well-known general solution of (18):

$$\xi(x,t,u) \equiv \xi(x,t) = \alpha_1 + \alpha_2 x + \alpha_3 t + \alpha_4 xt,$$
$$\tau(x,t,u) \equiv \tau(t) = 2\alpha_2 t + \alpha_4 t^2 + \alpha_5, \tag{19}$$
$$\eta(x,t,u) = [-\tfrac{1}{2}\alpha_3 x - \alpha_4(\tfrac{1}{4}x^2 + \tfrac{1}{2}t) + \alpha_6]u + g(x,t),$$

where $\alpha_1,\ldots,\alpha_6$ are arbitrary constants and, due to the linearity of PDE (17), $g(x,t)$ is an arbitrary solution of the heat equation, i.e., $g_t - g_{xx} = 0$.

The resulting invariant solutions of the heat equation appeared in B (1967) and B & Cole (1969).

Case 2. The nonclassical method: $\tau \equiv 1$

If $u = \Phi(x,t)$ satisfies the augmented PDE system $\mathbf{A}\{x;u\}$ consisting of the linear heat equation (17), the corresponding constraint invariant surface condition

$$u_t = \eta(x,t,u) - \xi(x,t,u)u_x, \tag{20}$$

and differential consequences of (20), it follows that all t-derivatives of u and all x-derivatives of u in the classical symmetry determining equation (18) can be as expressed as polynomials in u_x, with coefficients that are functions of x, t, and u.

In particular, after differentiating (20) with respect to x, and then replacing $u_{xx}(=u_t)$ by the right-hand side of (20), one obtains

$$u_{xt} = (\eta_x - \xi\eta) + (\eta_u - \xi_x + \xi^2)u_x - \xi_u u_x{}^2. \qquad (21)$$

After replacing u_t by the right-hand side of (20), and u_{xt} by the right-hand side of (21), the classical method determining equation (18) for infinitesimals $(\xi(x,t,u),\tau(x,t,u),\eta(x,t,u))$ becomes the nonclassical method determining equation for infinitesimals $(\xi(x,t,u),\eta(x,t,u))$:

$$\xi_{uu}u_x{}^3 + (2\xi_{xu} - \eta_{uu} - 2\xi\xi_u)u_x{}^2 + (\xi_{xx} - \xi_t - 2\eta_{xu} - 2\xi\xi_x + 2\eta\xi_u)u_x$$
$$+ (\eta_t - \eta_{xx} + 2\eta\xi_x) = 0. \qquad (22)$$

Equation (22) is a polynomial equation in u_x, and hence splits into four equations whose solution is given by

$$\begin{aligned} &\xi = \xi(x,t), \\ &\eta = C(x,t)u + D(x,t), \end{aligned} \tag{23}$$

where $\{\xi(x,t), C(x,t), D(x,t)\}$ is any solution of the ***nonlinear*** system

$$\begin{aligned} &\xi_t - \xi_{xx} + 2\xi\xi_x + 2C_x = 0, \\ &C_t - C_{xx} + 2\xi_x C = 0, \\ &D_t - D_{xx} + 2\xi_x D = 0. \end{aligned} \tag{24}$$

This case was considered in more detail in B & Cole (1969). Note that due to the form of (23), it follows that here all obtained solutions of the linear heat equation (17) are of the form (16).

Case 3. The nonclassical method: $\tau \equiv 0, \xi \equiv 1$

Here it is easy to show that after using the conditional invariant surface condition equation $u_x = \eta$ and its differential consequences, classical method determining equation (18) for infinitesimals $(\xi(x,t,u), \tau(x,t,u), \eta(x,t,u))$ becomes nonclassical method determining equation for infinitesimal $\eta(x,t,u)$:

$$\eta^2 \eta_{uu} + 2\eta\eta_{xu} + \eta_{xx} - \eta_t = 0. \quad (25)$$

Note that

$$\eta = -\frac{1}{2}\sigma(x,t)u \qquad (26)$$

solves the determining equation (25) if $v = \sigma(x,t)$ is any solution of the Burgers equation

$$v_t + vv_x - v_{xx} = 0. \qquad (27)$$

Thus equation (26) together with the conditional invariant surface condition equation $u_x = \eta$, yields the Hopf-Cole transformation

$$v = -2\frac{u_x}{u}$$

that relates solutions of the Burgers equation (27) and the linear heat equation (17) through the nonclassical method! This case was first considered in Fushchich et al. (1992).

Note that in this case, due to the form of an obtained infinitesimal that satisfies the determining equation (25), a resulting solution of the linear heat equation (17) is of the form

$$u = \phi(x, t, h(t)). \tag{28}$$

where $h(t)$ satisfies a reduced ODE.

(2) ***Boussinesq equation***

The nonclassical method essentially lay dormant for two decades. A significant discussion of it appeared in the papers of Olver & Rosenau [(1986), (1987)] in the context of finding solutions of PDEs subject to differential constraints.

A revived interest in the nonclassical method was ignited by the remarkable paper of Clarkson & Kruskal (1989), in which they exhibited solutions of the Boussinesq equation

$$u_{tt} + uu_{xx} + u_x^{\,2} + u_{xxxx} = 0, \qquad (29)$$

not obtainable by the classical method.

In this paper, the direct method was introduced to find solutions of the Boussinesq equation (29) that are of the form (15).

These were obtained by directly substituting the ansatz (15) into the Boussinesq equation (29) to find all cases leading to a reduced ODE for some $h(z(x,t))$.

In a "tour de force", Clarkson and Kruskal found all such solutions of (29). For example, their solutions of (29), given by

$$u(x,t) = t^2 h(z) - t^{-2}(x + \lambda t^5)^2, \quad z = xt + \tfrac{1}{6}\lambda t^6, \quad \lambda = \text{const}, \qquad (30)$$

with h(z) satisfying the reduced ODE

$$(w''' + ww' + 5\lambda w - 50\lambda^2 z)' = 0, \qquad (31)$$

were not obtainable by the classical method, i.e., as invariant solutions from the well-known point symmetries of the Boussinesq equation (29) with infinitesimals

$$\begin{aligned} \xi(x,t,u) &= \xi(x) = \alpha_1 x + \alpha_2, \\ \tau(x,t,u) &= \tau(t) = 2\alpha_1 t + \alpha_3, \\ \eta(x,t,u) &= \eta(u) = -2\alpha_1 u, \end{aligned} \qquad (32)$$

where $\alpha_1, \alpha_2, \alpha_3$, are arbitrary constants.

As shown above, ***all*** solutions arising from the direct method must arise from the nonclassical method.

In their seminal paper, Levi & Winternitz (1989) show how to use the nonclassical method to obtain all Clarkson and Kruskal solutions of the Boussinesq equation (29).

They consider the nonclassical method for the case $\tau \equiv 1$, and show that the resulting infinitesimals are:

$$\xi = \xi(x,t) = \alpha(t)x + \beta(t),$$
$$\eta(x,t,u) = -2\alpha(t)u - [2\alpha(t)(\alpha'(t) + 2\alpha^2(t))x^2 + 2([\alpha(t)\beta(t)]' + 4\alpha^2(t)\beta(t))x + 2\beta(t)(\beta'(t) + 2\alpha(t)\beta(t)], \tag{33}$$

where $\alpha(t)$ and $\beta(t)$ are solutions of ODE system

$$\begin{aligned} &\alpha'' + 2\alpha\alpha' - 4\alpha^3 = 0, \\ &\beta'' + 2\alpha\beta' - 4\alpha^2\beta = 0. \end{aligned} \tag{34}$$

The general solution of ODE system (34) is easily obtained and for any solution of ODE system (34), the general solution of the constraining invariant surface condition equation (12) is:

$$u = K^2(t)h(z) - (\alpha(t)x + \beta(t))^2, \quad z = K(t)x - \int_0^t \beta(s)K(s)ds, \tag{35}$$

where $K(t) = \exp\left[-\int_0^t \alpha(s)ds\right]$.

Substitution of (35) into Boussinesq equation (29) yields

$$w^{(4)} + ww'' + w'^2 + (Az + B)w' + 2Aw = 2(Az + B)^2 \qquad (36)$$

where

$$A = \frac{\alpha^2(t) - \alpha'(t)}{K^4(t)} = \text{const},$$

$$B = \frac{\alpha(t)\beta(t) - \beta'(t)}{K^3(t)} + A\int_0^t \beta(s)K(s)ds = \text{const}.$$

Note that the solutions obtained by the classical method result from the two particular sets of solutions

$$\alpha(t) = \frac{1}{2t + C},\ \beta(t) = \frac{D}{2t + C},\ C = \text{const},\ D = \text{const},$$

and $\alpha(t) = 0,\ \beta(t) = E = \text{const}$

of ODE system (34).

References

Bluman (1967) Construction of Solutions to Partial Differential Equations by the Use of Transformation Groups, Ph.D. Thesis, Caltech.

Bluman & Cole (1969) The general similarity solution of the heat equation *J. Math. Mech.* **18** 1025-1042.

Ovsiannikov (1962) *Group Properties of Differential Equations*, Nauka, Novosibirsk

Ovsiannikov (1982) *Group Analysis of Differential Equations*, Academic Press, New York

Bluman & Kumei (1989) *Symmetries and Differential Equations*, Springer, New York

Bluman & Anco (2002) *Symmetry and Integration Methods for Differential Equations*, Springer, New York

Clarkson & Kruskal (1989) New similarity solutions of the Boussinesq equation. *J. Math. Phys.* **30** 2201-2213

Levi & Winternitz (1989) Non-classical symmetry reduction: Example of the Boussinesq equation. *J. Phys.* **A22** 2915-2924

Nucci & Clarkson (1992) The nonclassical method is more general than the direct method for symmetry reductions. An example of the Fitzhugh-Nagumo equation. *Phys. Lett. A***164** 49-56

Fushchich, Shtelen, Serov, & Popovych (1992) Conditional symmetries of the linear heat equation. *Dokl. Ukr. Acad. Nauk A***12** 20-26.

Olver & Rosenau (1986) The construction of special solutions to partial differential equations. *Phys. Lett. A***114** 107-112.

Olver & Rosenau (1987) Group-invariant solutions of differential equations. *SIAM J. Appl. Math.* **47** 263-278.

Fokas & Liu (1994) Generalized conditional symmetries and exact solutions of nonintegrable equations *Teoret. Mat. Fiz.* **99** 263-277.