Invertible Mappings of Linear PDEs to Linear PDEs with Constant Coefficients

If a linear PDE has constant coefficients, then there is an arsenal of techniques (including use of Fourier series, Fourier and Laplace transforms) to find appropriate Green's functions and solve various posed boundary value problems. This leads to two obvious questions:

- Can one map a given linear PDE with variable coefficients to some linear PDE with constant coefficients by an invertible point transformation?

- What is the most general point transformation that yields such a mapping?

The second question is connected with the problem of finding all domains that yield the possibility of Fourier or Laplace transform analysis for a given linear PDE.

A constant coefficient linear PDE is completely characterized by its admitted point symmetries connected with its linearity and invariance under the Abelian group of translations of its independent variables.

Consequently, one can establish necessary and sufficient conditions for mapping a given variable coefficient linear PDE to some constant coefficient linear PDE.

When such conditions hold for a given linear PDE, one can find an explicit mapping.

Consider a given *p*th order linear PDE $\mathbf{R}\{x;u\}$ with n independent variables $x=(x^1,\ldots,x^n)$ and dependent variable u given by

$$a(x)u+a^i(x)u_i+\cdots+a^{i_1i_2\cdots i_p}(x)u_{i_1i_2\cdots i_p}=0,$$

defined on domain $D\subset\Re^n$. Aim is to determine whether or not $\mathbf{R}\{x;u\}$ can be mapped invertibly by some point transformation μ into some constant coefficient linear PDE $\mathbf{S}\{z;w\}$ of the form

$$bw+b^i w_i+\cdots+b^{i_1i_2\cdots i_p}w_{i_1i_2\cdots i_p}=0,$$

with independent variables $z=(z^1,\ldots,z^n)$ and dependent variable w, and find such a mapping μ when it exists.

In order to preserve linearity, such a mapping μ must be of the form

$$z^i = \phi^i(x), \quad i = 1, \ldots, n,$$
$$w = \psi(x, u) = G(x)u;$$

$G(x)$ is the *multiplier* of the mapping. The mapping μ is invertible if and only if

$$\det\left|\frac{\partial \phi^i}{\partial x^j}\right| \neq 0 \text{ in } D.$$

A constant coefficient linear PDE $\mathbf{S}\{z;w\}$ with n independent variables z is invariant under the n-parameter Lie group G_{zw} of translations of its independent variables.

Hence it is necessary that the given PDE $\mathbf{R}\{x;u\}$ admit an n-parameter Abelian Lie group G_{xu} to have an invertible mapping to a constant coefficient linear PDE.

Moreover G_{xu} must also be an n-parameter Abelian Lie group when its action is projected onto its space of n independent variables x since the mapping must preserve the commutation relations of the Abelian Lie algebra L_{zw}.

A constant coefficient linear PDE **S**$\{z;w\}$ with n independent variables admits:

$$\mathrm{Z}^{\alpha} = \frac{\partial}{\partial z^{\alpha}}, \quad \alpha = 1,\ldots,n.$$

Consequently, in order for the mapping μ to exist, the given linear PDE **R**$\{x;u\}$ must admit n infinitesimal generators of the form

$$\mathrm{X}^{\alpha} = \xi^{\alpha j}(x)\frac{\partial}{\partial x^{j}} + f^{\alpha}(x)u\frac{\partial}{\partial u}, \quad \alpha = 1,\ldots,n, \quad (1)$$

that satisfy the commutation relations

$$[\mathrm{X}^{\alpha},\mathrm{X}^{\beta}] = 0, \quad \alpha,\beta = 1,\ldots,n.$$

From the mapping equations

$$\mathrm{X}\phi = \mathrm{Z}z\big|_{(z,w)=(\phi,\psi)},$$
$$\mathrm{X}\psi = \mathrm{Z}w\big|_{(z,w)=(\phi,\psi)},$$

it follows that

$$\mathrm{X}^{\alpha}\phi^{i} = \mathrm{Z}^{\alpha}z^{i} = \delta^{\alpha i},$$
$$\mathrm{X}^{\alpha}(G(x)u) = \mathrm{Z}^{\alpha}w = 0, \quad \alpha, i = 1, \ldots, n;$$

More explicitly:

$$\begin{aligned} &\xi^{\alpha j}\frac{\partial \phi^{i}}{\partial x^{j}} = \delta^{\alpha i}, \quad \alpha, i = 1, \ldots, n; \\ &\xi^{\alpha j}\frac{\partial G}{\partial x^{j}} + f^{\alpha}G = 0, \quad \alpha = 1, \ldots, n. \end{aligned} \qquad (2)$$

Hence

$$\frac{\partial \phi^j}{\partial x^\alpha} \xi^{ji} = \delta^{\alpha i}, \quad \alpha, i = 1, \ldots, n.$$

Moreover, μ is invertible if and only if

$$\det | \xi^{ij}(x) | \neq 0 \quad \text{in} \quad D. \qquad (3)$$

From the commutation relations, it follows that for μ to exist, one must have:

$$\xi^{\beta k}\frac{\partial \xi^{\alpha j}}{\partial x^k} = \xi^{\beta k}\frac{\partial \xi^{\beta j}}{\partial x^k}, \qquad (4)$$

$$\xi^{\beta k}\frac{\partial f^{\alpha}}{\partial x^k} = \xi^{\beta k}\frac{\partial f^{\beta}}{\partial x^k}, \quad \alpha, \beta, j = 1, \ldots, n.$$

Theorem. If a given linear of PDE **R**$\{x;u\}$ admits n infinitesimal generators of the form (1) whose components $\{\xi^{ij}(x), f^i(x)\}$ satisfy equations (4) and condition (3), then there exists a solution $\{\phi^i(x), G(x)\}$ of equations (2) that defines an invertible mapping of **R**$\{x;u\}$ to some constant coefficient linear PDE **S**$\{z;w\}$.

Proof. The proof is accomplished by showing that any $\{\phi^i(x), G(x)\}$ solving (2), whose coefficients are defined by (3) and (4), satisfies the integrability conditions given by

$$\frac{\partial^2 \phi^i}{\partial x^j \partial x^k} = \frac{\partial^2 \phi^i}{\partial x^k \partial x^j}, \quad i, j, k = 1, \ldots, n;$$

$$\frac{\partial^2 G}{\partial x^j \partial x^k} = \frac{\partial^2 G}{\partial x^k \partial x^j}, \quad j, k = 1, \ldots, n.$$

Mapping algorithm summary

- Set up the determining equations for the infinitesimals of the point symmetries admitted by **R**$\{x;u\}$. [*Note that it is unnecessary to solve explicitly the determining equations*!]

- Use the determining equations to check if the coefficients of the linear equation **R**$\{x;u\}$ are such that (4) has a nontrivial solution for which $\det|\xi^{ij}(x)| \neq 0$ in some domain D. If the system of equations (4) only has trivial solutions for which $\det|\xi^{ij}(x)| \equiv 0$, then no invertible mapping μ exists.

- Solve the first set of equations (2) to find $\phi(x)$.

- Find the multiplier $G(x)$ by solving the second set of equations (2)

Example: Parabolic Equation

$$\frac{\partial^2 u'}{\partial x'^2} + \alpha(x', y')\frac{\partial u'}{\partial x'} + \beta(x', y')\frac{\partial u'}{\partial y'} + \gamma(x', y')u' = 0, \quad (5)$$

The point transformation $x = X(x', y') = \int^{x'} [\beta(t, y')]^{1/2} dt,\ y = y',$

$u = e^{-C(x,y)}u'$ where

$$C_x = -\tfrac{1}{2}[X_{y'} + \alpha(x', y')[\beta(x', y']^{-1/2} + \tfrac{1}{2}\beta^{-3/2}\beta_{x'}],$$

maps (5) to the standard form

$$u_{xx} + u_y + V(x, y)u = 0$$

with $V(x, y) = C_y + C_{xx} + C_x^2 + \frac{\gamma(x', y')}{\beta(x', y')}$.

Theorem 4.4.2-1. A parabolic PDE in standard form can be mapped invertibly by a point transformation to the backward heat equation

$$w_{z^1 z^1} + w_{z^2} = 0$$

if and only if $V(x, y)$ is of the form

$$V(x, y) = a(y)x^2 + b(y)x + c(y)$$

for some functions $a(y), b(y), c(y)$. The point transformation that yields the mapping is given by

$$z^1 = \sigma(y)x + \rho(y),$$

$$z^2 = \int^y \sigma^2(\hat{y})d\hat{y},$$

$$w = u \exp \tfrac{1}{4}[\sigma^{-1}\sigma'(y)x^2 + 2\sigma^{-1}\rho'(y)x + \lambda(y)],$$

where $(\sigma(y), \rho(y), \lambda(y))$ is a solution of the nonlinear system of ODEs

$$\begin{aligned} &\sigma^{-2}(\sigma\sigma'' - 2\sigma'^2) = 4a(y), \\ &(\sigma\rho'' - 2\sigma'\rho') = 2\sigma^2 b(y), \qquad (7) \\ &\lambda' = \sigma^{-2}(\rho'^2 - 2\sigma\sigma') + c(y). \end{aligned}$$

Equations (7) can be solved analytically.