Invertible Mappings of Nonlinear PDEs to Linear PDEs

Theorems on Invertible Mappings

Let $\mathbf{R}\{x;u\}$ denote a kth-order given system of N PDEs with n independent variables $x=(x^1,\dots,x^n)$ and m dependent variables $u=(u^1,\dots,u^m)$ given by

$$R^\sigma[u]=R^\sigma(x,u,\partial u,\dots,\partial^k u)=0, \quad \sigma=1,\dots,N.$$

Let $\mathbf{S}\{z;w\}$ denote a kth-order target system of N PDEs with n independent variables $z=(z^1,\dots,z^n)$ and m dependent variables $w=(w^1,\dots,w^m)$ given by

$$S^\sigma[w]=S^\sigma(z,w,\partial w,\dots,\partial^k w)=0, \quad \sigma=1,\dots,N.$$

Theorem 1 [case of one dependent variable u—due to Bäcklund (1876)]: *A mapping* μ *defines an invertible mapping from* $(x, u, \partial u, \ldots, \partial^p u)$*-space to* $(z, w, \partial w, \ldots, \partial^p w)$*-space for* **any** *fixed* $p \geq 1$ *if and only if* μ *is a* **one-to-one contact transformation** *of the form*

$$z = \phi(x, u, \partial u),$$
$$w = \psi(x, u, \partial u),$$
$$\partial w = \partial \psi(x, u, \partial u).$$

Theorem 2 [case of two or more dependent variables u—due to Müller and Matschat (1962)]: *A mapping* μ *defines an invertible mapping from*

$$(x, u, \partial u, \ldots, \partial^p u) - space\ to\ (z, w, \partial w, \ldots, \partial^p w) - space$$

for **any** *fixed* p *if and only if* μ *is a* **one-to-one point transformation** *of the form*

$$z = \phi(x, u),$$
$$w = \psi(x, u).$$

Theorem 3: *Suppose the target system of PDEs* $\mathbf{S}\{z;w\}$ *is* **completely characterized** *in terms of admitted point (contact) symmetries with infinitesimal generators*

$$\mathrm{Z} = \zeta^i(z,w,\partial w)\frac{\partial}{\partial z^i} + \omega^j(z,w,\partial w)\frac{\partial}{\partial w^j}.$$

Then the **necessary and sufficient conditions** *so that the given system* $\mathbf{R}\{x;u\}$ *can be mapped invertibly into a PDE system in the target system* $\mathbf{S}\{z;w\}$ *by some point (contact) transformation*

$z = \phi(x,u,\partial u),\ w = \psi(x,u,\partial u),$ *are that* $\mathbf{R}\{x,u\}$ *must admit point (contact) symmetries with infinitesimal generators*

$$\mathrm{X} = \xi^i(x,u,\partial u)\frac{\partial}{\partial x^i} + \omega^j(x,u,\partial u)\frac{\partial}{\partial u^j}$$

such that

$$\mathrm{X}\phi = \mathrm{Z}z\big|_{(z,w)=(\phi,\psi)},$$

$$\mathrm{X}\psi = \mathrm{Z}w\big|_{(z,w)=(\phi,\psi)}.$$

Invertible mappings of nonlinear PDEs to linear PDEs through admitted symmetries

Motivation: Here target system $\mathbf{S}\{z;w\}$ is some linear system (not known in advance), defined in terms of some (unknown) linear operator L[z], and given by

$$\mathrm{L}[z]w = g(z),$$

for some inhomogeneous term $g(z)$; $\mathbf{S}\{z;w\}$ is completely characterized by admitted infinite set of point symmetries

$$\mathrm{Z} = \omega \frac{\partial}{\partial w},$$

where $\omega = f(z)$ is any function satisfying $\mathrm{L}[z]f = 0$.

=> the following four theorems [Kumei & B (1984), B & Kumei (1990)]:

Theorem 4 (Necessary conditions for the existence of an invertible mapping for a nonlinear system of PDEs): *If there exists an invertible mapping* μ *of a given nonlinear system of PDEs* $\mathbf{R}\{x;u\}$, *with at least* $m = 2$ *dependent variables, to some linear system of PDEs* $\mathbf{S}\{z;w\}$, *then*

(1) *mapping is a point transformation of the form*

$$z^j = \phi^j(x,u), \quad j = 1,\dots,n,$$
$$w^\gamma = \psi^\gamma(x,u), \quad \gamma = 1,\dots,m;$$

(2) $\mathbf{R}\{x,u\}$ *admits an infinite set of point symmetries*

$$\mathrm{X} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\nu(x,u)\frac{\partial}{\partial u^\nu}$$

with infinitesimals of the form

$$\xi^i(x,u) = \sum_{\gamma=1}^{m} \alpha^i_\gamma(x,u)F^\gamma(x,u), \quad \eta^\nu(x,u) = \sum_{\gamma=1}^{m} \beta^\nu_\gamma(x,u)F^\gamma(x,u),$$

where $\alpha^i_\gamma(x,u)$, $\beta^\nu_\gamma(x,u)$, *are specific functions of* x *and* u, *and the components of* $F=(F^1,\ldots,F^m)$ *are arbitrary solutions of some linear system of PDEs*

$$\mathrm{L}[X]F = 0,$$

in terms of some specific linear differential linear operator $\mathrm{L}[X]$ *and specific independent variables*

$$X=(X^1(x,u),\ldots,X^n(x,u)).$$

Theorem 5 (Sufficient conditions for the existence of an invertible mapping for a nonlinear system of PDEs): *Suppose a given nonlinear system of PDEs* $\mathbf{R}\{x,u\}$, *with* $m \geq 2$ *dependent variables, admits an infinite set of point symmetries satisfying the criteria of Theorem* 4. *If the linear system of* m *first order PDEs for a scalar* Φ *given by*

$$\alpha_{i\sigma}(x,u)\frac{\partial \Phi}{\partial x_i} + \beta_\sigma^\nu(x,u)\frac{\partial \Phi}{\partial u^\nu} = 0, \quad \sigma = 1,\ldots,m,$$

has $X_1(x,u),\ldots,X_n(x,u)$ *as* n *functionally independent solutions, and the linear inhomogeneous system of* m^2 *first order PDEs*

$$\alpha_{i\sigma}(x,u)\frac{\partial \Psi^\gamma}{\partial x_i} + \beta_\sigma^\nu(x,u)\frac{\partial \Psi^\gamma}{\partial u^\nu} = \delta_\sigma^\gamma,$$

where δ_σ^γ *is the Kronecker symbol,* γ, $\sigma = 1,\ldots,m$, *has some particular solution*

$$\Psi = (\psi^1(x,u),\ldots,\psi^m(x,u)),$$

then the mapping μ given by

$$z^j = \phi^j(x,u) = X^j(x,u), \quad j = 1, \ldots, n,$$
$$w^\gamma = \psi^\gamma(x,u), \quad \gamma = 1, \ldots, m,$$

is invertible and transforms $\mathbf{R}\{x;u\}$ *to the linear system of PDEs* $\mathbf{S}\{z;w\}$ *given by*

$$\mathrm{L}[z]w = g(z),$$

for some inhomogeneous term $g(z)$.

Theorem 6 (Necessary conditions for the existence of an invertible mapping for a nonlinear scalar PDE): *If there exists an invertible mapping* μ *of a given nonlinear scalar PDE* $\mathbf{R}\{x;u\}$ *to some linear scalar PDE* $\mathbf{S}\{z;w\}$, *then*

(1) *the mapping is a contact transformation of the form*

$$\begin{aligned} z^j &= \phi^j(x,u,\partial u), \\ w &= \psi(x,u,\partial u), \\ \frac{\partial w}{\partial z^j} &= \partial\psi^j(x,u,\partial u), \quad j=1,\ldots,n; \end{aligned}$$

(2) $\mathbf{R}\{x;u\}$ *admits an infinite set of contact symmetries*

$$\mathrm{X} = \xi^i(x,u,\partial u)\frac{\partial}{\partial x^i} + \eta(x,u,\partial u)\frac{\partial}{\partial u} + \eta^{(1)i}(x,u,\partial u)\frac{\partial}{\partial u^i}$$

with infinitesimals $\xi^i(x,u,\partial u), \eta(x,u,\partial u),$ *and first extended infinitesimals* $\eta_i^{(1)}(x,u,\partial u)$ *of the form*

$$\xi^i(x,u,\partial u) = \alpha^i(x,u,\partial u)F(x,u,\partial u) + \alpha^i_j(x,u,\partial u)H^j(x,u,\partial u),$$

$$\eta(x,u,\partial u) = \beta(x,u,\partial u)F(x,u,\partial u) + \beta_j(x,u,\partial u)H^j(x,u,\partial u),$$

$$\eta^{(1)i}(x,u,\partial u) = \lambda^i(x,u,\partial u)F(x,u,\partial u) + \lambda^i_j(x,u,\partial u)H^j(x,u,\partial u),$$

where α^i, $\alpha^i{}_j$, β, β_j, λ^i, $\lambda^i{}_j$, $i, j = 1,\ldots,n$, *are specific functions of* x, u, *and the*

components of ∂u, *and* $F(x,u,\partial u)$ *is an arbitrary solution of some linear scalar PDE* $\mathrm{L}[X]F = 0$, *in terms of some specific linear differential operator* $\mathrm{L}[X]$ *and specific independent variables*

$X = (X^1(x,u,\partial u),\ldots,X^n(x,u,\partial u))$; *and* $H^j(x,u,\partial u)$ *satisfies*

$$H^j = \frac{\partial F}{\partial X^j}, \quad j = 1,\ldots,n.$$

Theorem 7 (Sufficient conditions for the existence of an invertible mapping for a nonlinear scalar PDE): *Suppose a given nonlinear scalar PDE* $\mathbf{R}\{x;u\}$, *admits an infinite set of contact symmetries satisfying the criteria of Theorem* 6. *Suppose the following conditions hold.*

(1) *The linear homogeneous system of* $n+1$ *first order PDEs for a scalar*

$\Phi(x,u,\partial u)$ *given by*

$$\alpha^i \frac{\partial\Phi}{\partial x^i} + \beta \frac{\partial\Phi}{\partial u} + \lambda^i \frac{\partial\Phi}{\partial u^i} = 0,$$
$$\alpha^i_j \frac{\partial\Phi}{\partial x^i} + \beta_j \frac{\partial\Phi}{\partial u} + \lambda^i_j \frac{\partial\Phi}{\partial u^i} = 0, \quad j = 1,\ldots,n,$$

has $X^1(x,u,\partial u),\ldots,X^n(x,u,\partial u)$ *as* n *functionally independent solutions.*

(2) *The linear inhomogeneous system of* $n+1$ *first order PDEs*

$$\alpha^i \frac{\partial \Psi}{\partial x^i} + \beta \frac{\partial \Psi}{\partial u} + \lambda^i \frac{\partial \Psi}{\partial u^i} = 1,$$

$$\alpha^i_j \frac{\partial \Psi}{\partial x^i} + \beta_j \frac{\partial \Psi}{\partial u} + \lambda^i_j \frac{\partial \Psi}{\partial u^i} = 0, \quad j = 1, \ldots, n,$$

has some particular solution $\Psi = \psi(x, u, \partial u)$.

(3) *The linear inhomogeneous system of* $n(n+1)$ *first order PDEs*

$$\alpha^i \frac{\partial \Psi^j}{\partial x^i} + \beta \frac{\partial \Psi^j}{\partial u} + \lambda^i \frac{\partial \Psi^j}{\partial u^i} = 1,$$

$$\alpha^i_k \frac{\partial \Psi^j}{\partial x^i} + \beta_k \frac{\partial \Psi^j}{\partial u} + \lambda^i_k \frac{\partial \Psi^j}{\partial u^i} = \delta^j_k, \quad j, k = 1, \ldots, n, \tag{*}$$

where δ^j_k *is the Kronecker symbol, has some particular solution*

$$(\Psi^1, \ldots. \Psi^n) = \partial \psi = (\psi^1(x, u, \partial u), \ldots, \psi^n(x, u, \partial u)).$$

(4) *There exists a particular solution* $\partial\psi$ *of* (*) *such that* $(z,w,\partial w) = (X(x,u,\partial u),\psi(x,u,\partial u),\partial\psi(x,u,\partial u))$ *defines a contact transformation.*

Then the mapping μ *given by*

$$
\begin{aligned}
&z^j = \phi^j(x,u,\partial u) = X^j(x,u,\partial u),\\
&w = \psi(x,u,\partial u),\\
&w^j = \psi^j(x,u,\partial u), \quad j = 1,\ldots,n,
\end{aligned}
$$

is invertible and transforms $\mathbf{R}\{x;u\}$ *to the linear scalar PDE* $\mathbf{S}\{z;w\}$ *given by*

$$\mathrm{L}[z]w = g(z),$$

for some inhomogeneous term $g(z)$.

Invertible mappings of nonlinear PDEs to linear PDEs through admitted conservation law multipliers

Definition : The set of factors $\{\Lambda_\nu[U]\}$ is a *set of conservation law multipliers* for a system of PDEs $\mathbf{R}\{x;u\}$ if and only if for **arbitrary** functions $U(x) = (U^1(x),\ldots, U^m(x))$, one has an identity (divergence expression)

$$\Lambda_\nu[U]R^\nu[U] \equiv \mathrm{D}_i\Phi^i[U]$$

holding for some functions $\Phi^i[U]$, $i = 1,\ldots,n$; D_i are total derivative operators.

Motivation: Here target system $\mathbf{S}\{z;w\}$ is some linear system (not known in advance), defined in terms of some (unknown) linear operator L[z]. For any linear operator L[z] and its adjoint operator L*[z], the formal relation

$$V\mathrm{L}[z]W - W\mathrm{L}^*[z]V$$

is a divergence expression for arbitrary functions $V(z)$, $W(z)$.

Consider a kth order linear PDE system $\mathrm{L}[z]w=0,$ denoted by $\mathbf{S}\{z;w\}.$
In particular, the linear PDEs are given by

$$\mathrm{L}_\alpha^\sigma[z]w^\alpha=0, \qquad (*)$$

in terms of linear operators

$$\mathrm{L}_\alpha^\sigma[z]=b_\alpha^\sigma(z)+b_\alpha^{\sigma i}(z)\frac{\partial}{\partial z^i}+\cdots+b_\alpha^{\sigma i_1\cdots i_k}(z)\frac{\partial^k}{\partial z^{i_1}\cdots\partial z^{i_k}}, \quad \sigma=1,\dots,N.$$

The corresponding **adjoint linear system**, $\mathrm{L}^*[z]v = 0$, is given by

$$\mathrm{L}^{*\sigma}_{\ \alpha}[z]v_\sigma=0, \qquad (\dagger)$$

where for any functions $V(z)=(V_1(z),\dots,V_N(z)),$

$$\mathrm{L}^{*\sigma}_{\ \alpha}[z]V_\sigma=b_\alpha^\sigma(z)V_\sigma-\frac{\partial}{\partial z^i}(b_\alpha^{\sigma i}(z)V_\sigma)+\cdots$$

$$+(-1)^k\frac{\partial^k}{\partial z^{i_1}\cdots\partial z^{i_k}}(b_\alpha^{\sigma i_1\cdots i_k}(z)V_\sigma), \quad \alpha=1,\dots,m.$$

Then for arbitrary functions $V(z)$ and $W(z) = (W^1(z),\ldots,W^m(z))$, which one can view as **conservation law multipliers** $\{V_\sigma(z)\}$ and $\{W^\alpha(z)\}$ for the **augmented linear system** consisting of the linear system (*) and the adjoint system (†), one has a conservation law identity

$$\mathrm{D}\Psi^i / \mathrm{D}z^i = V_\sigma \mathrm{L}^\sigma_\alpha[z]W^\alpha - W^\alpha \mathrm{L}^{*\sigma}_\alpha[z]V_\sigma,$$

holding for some specific functions $\{\Psi^i(z)\}$ that have a bilinear dependence on the multipliers and their derivatives.

Remark: Suppose a given nonlinear system of PDEs $\mathbf{R}\{x;u\}$ can be invertibly mapped to some linear system of PDEs $\mathbf{S}\{z;w\}$ by some point transformation. Then for some nontrivial factors $\{Q_\nu^\sigma[U]\}$, one must have

$$Q_\nu^\sigma[U]R^\nu[U] = \mathrm{L}_\alpha^\sigma[z]W^\alpha, \quad \sigma = 1,\dots,N,$$

where $U(x) = (U^1(x),\dots,U^m(x))$ are arbitrary functions and functions $W(z) = (W^1(z),\dots,W^m(z)),$ are obtained through the linearization mapping

$$\begin{aligned} z &= \phi(x,U(x)), \\ W(z) &= \psi(x,U(x)). \end{aligned}$$

Consequently, the CL identity

$$D\Psi^i / Dz^i = V_\sigma L^\sigma_\alpha[z] W^\alpha - W^\alpha L^{*\sigma}_\alpha[z] V_\sigma$$

becomes

$$D\lambda^i / Dx^i = J(V_\sigma Q^\sigma_\nu[U] R^\nu[U] - W^\alpha L^{*\sigma}_\alpha[z] V_\sigma),$$

where $D\lambda^i / Dx^i = |Dz / Dx| (D\Psi^i / Dz^i)$ in terms of some specific functions λ^i and the Jacobian factor

$$J = \left|\frac{Dz}{Dx}\right| = \det\left(\frac{Dz^i}{Dx^j}\right).$$

This leads to the following two theorems.

Theorem 8 (Necessary conditions for the existence of an invertible mapping). *If there exists an invertible point transformation that maps a given kth order nonlinear system of PDEs* $\mathbf{R}\{x;u\}$ *to some linear system of PDEs* $\mathbf{S}\{z;w\}$, *then the nonlinear system* $\mathbf{R}\{x;u\}$ *must admit an infinite set of conservation law multipliers of the form*

$$\Lambda_\nu[U] = J v_\sigma Q_\nu^\sigma[U],$$

where $Q_\nu^\sigma[U], \nu = 1,2,\ldots,N, \sigma = 1,2,\ldots,N,$ *are specific functions of* x *and* U *and derivatives of* U *to order* $k-1$, *and the components of* $v = (v_1,\ldots,v_m)$ *are dependent variables of some linear system of PDEs* $\tilde{\mathrm{L}}[X]v = 0$, *given by*

$$\tilde{\mathrm{L}}_\alpha^\sigma[X]v_\sigma = 0, \quad \alpha = 1,2,\ldots,m,$$

in terms of specific independent variables

$$X = (X^1(x,U),\ldots,X^n(x,U)),$$

and

$$J = |\mathrm{D}X/\mathrm{D}x|.$$

Theorem 9 (Sufficient conditions for the existence of an invertible mapping). *Suppose a given nonlinear system of PDEs* $\mathbf{R}\{x;u\}$ *admits an infinite set of conservation law multipliers satisfying the criteria of Theorem* 8. *Let* $\tilde{\mathrm{L}}^{*}[X]$ *be the adjoint of the linear operator* $\tilde{\mathrm{L}}[X]$. *Consider the augmented system of PDEs consisting of the given nonlinear system of PDEs* $\mathbf{R}\{x;u\}$ *and the linear system* $\tilde{\mathrm{L}}[X]v = 0$. *Then there exist multipliers*

$$\{\Lambda_\nu = JV_\sigma Q_\nu^\sigma[U], -JW^\alpha(x,U)\}$$

so that a conservation law identity

$$\Lambda_\nu R^\nu[U] - JW^\alpha(x,U)\tilde{\mathrm{L}}_\alpha^\sigma[X(x,U)]V_\sigma = \mathrm{D}\Theta^i / \mathrm{D}x^i$$

holds for some specific functions $\Theta^i(x)$, *where the Jacobian*

$$J = |\mathrm{D}X / \mathrm{D}x| = \det(\mathrm{D}X^i / \mathrm{D}x^j).$$

Then the identity $\Lambda_\nu R^\nu[U] - JW^\alpha(x,U)\tilde{\mathrm{L}}^\sigma_\alpha[X(x,U)]V_\sigma = \mathrm{D}\Theta^i / \mathrm{D}x^i$ *becomes*

$$V_\sigma Q^\sigma_\nu[U]R^\nu[U] - W^\alpha(x,U)\tilde{\mathrm{L}}^\sigma_\alpha[X(x,U)]V^\sigma = \mathrm{D}\Gamma^i / \mathrm{D}X^i,$$

for some functions Γ^i. *Consequently, the point transformation given by*

$$z = X(x,u),\ w = W(x,u)$$

maps the nonlinear system of PDEs $\mathbf{R}\{x;u\}$ *invertibly into the linear system given by*

$$\tilde{\mathrm{L}}^{*\sigma}_\alpha[z]w^\alpha = 0,$$

provided that this point transformation is an invertible transformation.

Proof. Since $\widetilde{\mathrm{L}}[X]$ is a linear operator, it follows that the conservation law identity

$$W^{\alpha}(x,U)\widetilde{\mathrm{L}}_{\alpha}^{\sigma}[X(x,U)]V_{\sigma} = V_{\sigma}\widetilde{\mathrm{L}}^{*\sigma}_{\alpha}[X(x,U)]W^{\alpha}(x,U) + \mathrm{D}\theta^{i}/\mathrm{D}X^{i}$$

holds for some specific functions $\theta^{i}[U,V,W]$. Consequently, the identity

$$V_{\sigma}Q_{\nu}^{\sigma}[U]R^{\nu}[U] - W^{\alpha}(x,U)\widetilde{\mathrm{L}}_{\alpha}^{\sigma}[X(x,U)]V_{\sigma} = \mathrm{D}\Gamma^{i}/\mathrm{D}X^{i},$$

becomes

$$V_{\sigma}(Q_{\nu}^{\sigma}[U]R^{\nu}[U] - \widetilde{\mathrm{L}}^{*\sigma}_{\alpha}[X(x,U)]W^{\alpha}(x,U)) = \mathrm{D}(\Gamma^{i}+\theta^{i})/\mathrm{D}X^{i}. \quad (1)$$

Now apply the Euler operators with respect to V_{σ}, i.e.,

$$E_{V_{\sigma}} = \frac{\partial}{\partial V_{\sigma}} - \frac{\mathrm{D}}{\mathrm{D}X^{i}}\frac{\partial}{\partial(\frac{\partial V_{\sigma}}{\partial X^{i}})} + \cdots, \ \sigma = 1,\ldots,m,$$

to each side of equation (1). Each such Euler operator annihilates the r.h.s of (1).

Consequently, one obtains the identity

$$Q_{\nu}^{\sigma}[U]R^{\nu}[U] = \tilde{L}^{*\sigma}_{\alpha}[X(x,U)]W^{\alpha}(x,U)$$

holding for arbitrary functions U. Now suppose $U = u$ solves the given nonlinear system of **R**$\{x;u\}$. Then it follows that $w = W(x,u)$ solves the linear system

$$\tilde{L}^{*\sigma}_{\alpha}[z]w^{\alpha} = 0. \qquad (2)$$

=> point transformation $z = X(x,u)$, $w = W(x,u)$.

Next check that this point transformation is an invertible transformation. If yes, then it invertibly maps the nonlinear system of PDEs **R**$\{x;u\}$ into the linear system (2). QED

Remark: Theorems 8 and 9 are easily modified to include mappings of nonlinear scalar PDEs to linear PDEs through contact transformations.

Examples of Linearizations of Nonlinear PDEs Through Admitted Symmetries and Through Admitted Conservation Law Multipliers

LINEARIZATION OF BURGERS' EQUATION

Consider $\mathbf{R}\{x;u\}$ with independent variables $(x^1, x^2) = (x, t)$ and dependent variables (u^1, u^2), given by the system

$$R^1[u] = \frac{\partial u^2}{\partial x} - 2u^1 = 0,$$

$$R^2[u] = \frac{\partial u^2}{\partial t} - 2\frac{\partial u^1}{\partial x} + (u^1)^2 = 0.$$

Then $u^1 = u$ satisfies Burgers' equation

$$u_{xx} - uu_x - u_t = 0.$$

Linearization Through Admitted Point Symmetries

Burgers' equation admits at most a finite number of contact symmetries. **Hence there exists no point or contact transformation that linearizes Burgers' equation.**

But the nonlinear system **R**{x;u} admits an infinite set of point symmetries represented by the infinitesimal generator [Krasil'shchik & Vinogradov (1984)]

$$\mathrm{X} = e^{u^2/4}\left\{[2h(x,t) + g(x,t)u^1]\frac{\partial}{\partial u^1} + 4g(x,t)\frac{\partial}{\partial u^2}\right\}$$

where $(g(x,t), h(x,t))$ is an arbitrary solution of the linear system of PDEs

$$h = g_x, \quad h_x = g_t.$$

=> one can linearize **R**{x;u} by an invertible mapping.

Then

$$F^1 = h(x,t),\ F^2 = g(x,t),\ \alpha_j^i = 0,$$

$$\beta_1^1 = 2e^{u^2/4},\ \beta_2^1 = u^1 e^{u^2/4},\ \beta_1^2 = 0,\ \beta_2^2 = 4e^{u^2/4}.$$

$$\Rightarrow X^1 = x,\ X^2 = t$$

The corresponding linear inhomogeneous system has as a particular solution

$$\Psi = (\psi^1, \psi^2) = (\tfrac{1}{2} u^1 e^{-u^2/4}, -e^{-u^2/4}).$$

⇨ invertible mapping

$$z^1 = x,\ z^2 = t,\ w^1 = \tfrac{1}{2} u^1 e^{-u^2/4},\ w^2 = -e^{-u^2/4}$$

from **R**$\{x;u\}$ to linear system **S**$\{z;w\}$

$$w^1 = \frac{\partial w^2}{\partial x}, \quad \frac{\partial w^1}{\partial x} = \frac{\partial w^2}{\partial t}.$$

Note that w^1 satisfies the heat equation

$$\frac{\partial^2 w^1}{\partial x^2} - \frac{\partial w^1}{\partial t} = 0.$$

⇨ (non-invertible) Hopf-Cole transformation

$$u = u^1 = -\frac{2}{w^1}\frac{\partial w^1}{\partial x}.$$

Linearization Through Admitted Conservation Law Multipliers

The nonlinear system $\mathbf{R}\{x;u\}$ admits an infinite set of conservation law multipliers of the form $\Lambda_i[U] = \Lambda_i(x,t,U)$ given by

$$\Lambda_1[U] = v_1(\tfrac{1}{2}U^1 e^{-U^2/4}) + v_2 e^{-U^2/4}, \quad \Lambda_2[U] = v_1 e^{-U^2/4},$$

where $(v_1(x,t), v_2(x,t))$ is any solution of the linear system

$$\frac{\partial v_1}{\partial x} - v_2 = 0, \quad \frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial t} = 0.$$

Hence, the necessary conditions for the existence of an invertible mapping of the nonlinear system $\mathbf{R}\{x;u\}$ to a linear system are satisfied, where the target linear system has the same independent variables as the given system.

In the conservation law arising from the infinite set of multipliers $\Lambda_1[U] = v_1(\frac{1}{2}U^1 e^{-U^2/4}) + v_2 e^{-U^2/4}, \Lambda_2[U] = v_1 e^{-U^2/4}$, replace (v_1, v_2) by arbitrary functions (V_1, V_2).

=> conservation law identity for the augmented system, consisting of the given nonlinear system $\mathbf{R}\{x;u\}$ and the linear system

$$\frac{\partial v_1}{\partial x} - v_2 = 0, \frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial t} = 0 : \qquad (*)$$

$$\begin{aligned}
&[V_1(\tfrac{1}{2}U^1 e^{-U^2/4}) + V_2 e^{-U^2/4}]R^1[U] + V_1 e^{-U^2/4} R^2[U] \\
&\qquad - 2U^1 e^{-U^2/4}\left[\mathrm{D}V_1/\mathrm{D}x - V_2\right] - 4e^{-U^2/4}\left[\mathrm{D}V_2/\mathrm{D}x + \mathrm{D}V_1/\mathrm{D}t\right] \\
&\qquad = \frac{\mathrm{D}}{\mathrm{D}x}\left[e^{-U^2/4}(-4V_2 - 2U^1 V_1)\right] + \frac{\mathrm{D}}{\mathrm{D}t}\left[-4V_1 e^{-U^2/4}\right].
\end{aligned}$$

Consequently, the sufficiency conditions of Theorem 9 yield an invertible mapping of the nonlinear system $\mathbf{R}\{x;u\}$ to a linear system which is the adjoint of the linear system (*). In particular, this yields the same mapping obtained from the admitted infinite set of point symmetries.

LINEARIZATION OF A PIPELINE FLOW EQUATION

Let $\mathbf{R}\{x;u\}$ be the pipeline flow equation

$$R[u] = u_t u_{xx} + u_x{}^p = 0.$$

Linearization Through Admitted Contact Symmetries

$\mathbf{R}\{x;u\}$ admits an infinite set of contact symmetries represented by the infinitesimal generator

$$\mathrm{X} = -\frac{\partial F}{\partial u_x}\frac{\partial}{\partial x} + \left[F - u_x \frac{\partial F}{\partial u_x} \right]\frac{\partial}{\partial u} + \frac{\partial F}{\partial t}\frac{\partial}{\partial u_t}$$

where $F(x,u,\partial u) = F(t,u_x)$ is any solution of the linear PDE

$$u_x^p \frac{\partial^2 F}{\partial u_x^2} - \frac{\partial F}{\partial t} = 0.$$

⇨ one can linearize $\mathbf{R}\{x;u\}$ by a contact transformation. Here

$$X^1 = u_x,\ X^2 = t,\ \alpha^i = 0,\ \alpha_1^1 = -1,\ \alpha_1^2 = \alpha_2^1 = \alpha_2^2 = 0,\ \beta = 1,\ \beta_1 = -u_x,\ \beta_2 = 0,\ \lambda^i = 0,$$
$$\lambda_1^1 = \lambda_2^1 = \lambda_1^2 = 0,\ \lambda_2^2 = 1.\ \beta_1^1 = 2e^{u^2/4},\ \beta_2^1 = u^1 e^{u^2/4},\ \beta_1^2 = 0,\ \beta_2^2 = 4e^{u^2/4}.$$

In Theorem 7, the corresponding linear homogeneous system has

$$X^1 = u_x,\ X^2 = t$$

as functionally independent solutions; the corresponding linear homogeneous system has a particular solution

$$\psi = u - xu_x,$$

and the corresponding linear inhomogeneous system has as a particular solution

$$(\psi^1, \psi^2) = (u_t, -x).$$

=> invertible mapping μ given by the contact transformation

$$z^1 = t, \quad z^2 = u_x, \quad w = u - xu_x, \quad w^1 = u_t, \quad w^2 = -x,$$

transforms the nonlinear PDE $\mathbf{R}\{x,u\}$ to the linear PDE

$$u_x^p \frac{\partial^2 w}{\partial {u_x}^2} - \frac{\partial w}{\partial t} = 0.$$

Linearization Through Admitted Conservation Law Multipliers

One can show that the scalar nonlinear PDE $\mathbf{R}\{x;u\}$ admits an infinite set of conservation law multipliers of the form $\Lambda[U]=\Lambda(x,t,U,U_x,U_t)$ given by

$$\Lambda[U]=v(X^1,X^2)=v(U_x,t),$$

where $v(X^1,X^2)$ is any solution of the linear PDE

$$\frac{\partial v}{\partial X^2}+\frac{\partial^2((X^1)^p v)}{\partial X^{1^2}}=0.$$

Hence the necessary conditions for the existence of an invertible contact transformation that linearizes $\mathbf{R}\{x;u\}$ are satisfied with a target linear system having as independent variables

$$X^1=u_x,\ X^2=t$$

In the conservation law arising from the infinite set of multipliers $\Lambda[U] = v(X^1, X^2) = v(U_x, t)$, replace v by an arbitrary function V.

⇨ conservation law identity for the augmented system, consisting of **R**{x;u} and linear PDE

$$\frac{\partial v}{\partial X^2} + \frac{\partial^2((X^1)^p v)}{\partial {X^1}^2} = 0 : \quad (*)$$

$$VR[U] - J(xU_x - U)\left[\frac{\partial V}{\partial X^2} + \frac{\partial^2((X^1)^p V)}{\partial {X^1}^2}\right]$$
$$= \frac{\mathrm{D}}{\mathrm{D}x}\left[(xU_x - U)(U_{tx}V + {U_x}^p V_{X^1}) + ((1-p)xU_x + pU){U_x}^{p-1}V\right]$$
$$+ \frac{\mathrm{D}}{\mathrm{D}t}\left[U_{xx}(U - xU_x)V\right],$$

where Jacobian

$$J = \left|\frac{\mathrm{D}X}{\mathrm{D}x}\right| = \det\begin{bmatrix} U_{xx} & U_{xt} \\ 0 & 1 \end{bmatrix} = U_{xx}.$$

In verifying the conservation law identity (4.20), note that

$$V_x = V_{X^1} U_{xx}, \quad V_t = V_{X^1} U_{xt} + V_{X^2}.$$

⇨ sufficiency conditions of modified Theorem 9 holding for the existence of an invertible mapping by a contact transformation of $\mathbf{R}\{x;u\}$ to a linear PDE which is the adjoint of linear PDE (*).

⇨ invertible contact transformation given by

$$X^1 = u_x, \; X^2 = t, \; w = xu_x - u, \; w_{X^1} = x, \; w_{X^2} = -u_t,$$

maps the nonlinear pipeline equation given $\mathbf{R}\{x;u\}$ into the linear PDE

$$(X^1)^p \frac{\partial^2 w}{\partial X^{1^2}} - \frac{\partial w}{\partial X^2} = 0$$

which is the adjoint of the linear PDE (*).

LINEARIZATION OF A NONLINEAR TELEGRAPH EQUATION

Let $\mathbf{R}\{x;u\}$ be the nonlinear telegraph (NLT) system, given by

$$R^1[u] = \frac{\partial u^2}{\partial t} - \frac{\partial u^1}{\partial x} = 0,$$
$$R^2[u] = \frac{\partial u^1}{\partial t} + u^1(u^1 - 1) - (u^1)^2 \frac{\partial u^2}{\partial x} = 0.$$

Linearization Through Admitted Point Symmetries

The NLT system admits an infinite set of point symmetries represented by the infinitesimal generator

$$\mathrm{X} = F^1(X,T)\frac{\partial}{\partial x} + e^{-t}F^2(X,T)\frac{\partial}{\partial t} + e^{-t}u^1F^2(X,T)\frac{\partial}{\partial u^1} + F^1(X,T)\frac{\partial}{\partial u^2}$$

where

$$X^1 = X = x - u^2, \quad X^2 = T = t - \log u^1,$$

and $(F^1(X,T), F^2(X,T))$ is an arbitrary solution of the linear system of PDEs

$$\frac{\partial F^2}{\partial T} - e^T \frac{\partial F^1}{\partial X} = 0,$$

$$\frac{\partial F^2}{\partial X} - e^T \frac{\partial F^1}{\partial T} = 0.$$

=> one can linearize $\mathbf{R}\{x;u\}$ by an invertible mapping.

Then $\alpha_1^1 = \beta_1^2 = 1, \alpha_2^1 = \alpha_1^2 = \beta_1^1 = \beta_2^2 = 0, \alpha_2^2 = e^{-t}, \beta_2^1 = e^{-t}u.$

The corresponding linear inhomogeneous system has as a particular solution $\Psi = (\psi^1, \psi^2) = (x, e^t).$

Hence the invertible mapping given by

$$z^1 = x - v, \quad z^2 = t - \log u, \quad w^1 = x, \quad w^2 = e^t,$$

transforms the NLT system to the linear PDE $\mathbf{S}\{z;w\}$ given by

$$\frac{\partial w^2}{\partial z^2} - e^{z^2}\frac{\partial w^1}{\partial z^1} = 0,$$

$$\frac{\partial w^2}{\partial z^1} - e^{z^2}\frac{\partial w^1}{\partial z^2} = 0.$$

Linearization Through Admitted Conservation Law Multipliers

NLT system admits an infinite set of conservation law multipliers of the form $\Lambda_i[U]=\Lambda_i(x,t,U^1,U^2)$: After some integrability analysis, one obtains

$$\Lambda_1[U]=f_{U^2},\quad \Lambda_2[U]=f_{U^1} \qquad (*)$$

in terms of $f(x,t,U^1,U^2)$ satisfying

$$f_x+f_{U^2}=0,\quad f_t+U^1f_{U^1}=0,\quad (U^1)^2f_{U^1U^1}-2U^1f_{U^1}-f_{U^2U^2}=0. \qquad (**)$$

The solution of the first two PDEs of (**) yields $f = f(X,T)$ where

$$X=x-U^2,\, T=t-\log U^1,$$

and then the third PDE of (**) combined with (*) yield the infinite set of multipliers

$$\Lambda_1[U]=v_1(X,T)\quad \Lambda_2[U]=v_2(X,T)(U^1)^{-1},$$

where $(v_1(X,T),v_2(X,T))$ is any solution of the linear system

$$\frac{\partial v_1}{\partial X}-\frac{\partial v_2}{\partial T}+v_2=0,$$

$$\frac{\partial v_2}{\partial X}-\frac{\partial v_1}{\partial T}=0.$$

Hence, the necessary conditions for the existence of an invertible mapping of the nonlinear NLT system $\mathbf{R}\{x;u\}$ to a linear system are satisfied.

In the conservation law arising from this finite set of multipliers, replace (v_1,v_2) by arbitrary functions (V_1,V_2).

⇨ conservation law identity for the augmented system, consisting of the given nonlinear NLT system and the linear system

$$\frac{\partial v_1}{\partial X}-\frac{\partial v_2}{\partial T}+v_2=0, \quad \frac{\partial v_2}{\partial X}-\frac{\partial v_1}{\partial T}=0:$$

$$V_1R^1[U]+V_2(U^1)^{-1}R^2[U]-JU^1\left[\mathrm{D}_XV_1-\mathrm{D}_TV_2+V_2\right]+Jx\left[\mathrm{D}_XV_2-\mathrm{D}_TV_1\right]$$
$$=\mathrm{D}_x\left[-V_1\left(x\frac{\partial U^2}{\partial t}-U^1+\frac{\partial U^1}{\partial t}\right)+V_2\left(x-x(U^1)^{-1}\frac{\partial U^1}{\partial t}-U^1\frac{\partial U^2}{\partial t}\right)\right]$$
$$+\mathrm{D}_t\left[-V_1\left(x-x\frac{\partial U^2}{\partial x}+\frac{\partial U^1}{\partial t}\right)+V_2\left(x(U^1)^{-1}\frac{\partial U^1}{\partial x}-U^1+U^1\frac{\partial U^2}{\partial x}\right)\right],$$

with $J=(U^1)^{-1}\left[\left(1-\frac{\partial U^2}{\partial x}\right)\left(U^1-\frac{\partial U^1}{\partial t}\right)-\frac{\partial U^2}{\partial t}\frac{\partial U^1}{\partial x}\right].$

$$\Rightarrow \qquad z^1 = X = x - u^2,\ z^2 = T = t - \log u^1,\ w^1 = -x,\ w^2 = u^1,$$

maps the NLT system into the linear system

$$\frac{\partial w^1}{\partial X} - \frac{\partial w^2}{\partial T} - w^2 = 0,$$
$$\frac{\partial w^2}{\partial X} - \frac{\partial w^1}{\partial T} = 0, \qquad (1)$$

Note that the point transformation

$$\tilde{w}^1 = w^1, \quad \tilde{w}^2 = e^T w^2,$$

maps the linear system (1) into the linear system

$$\frac{\partial \tilde{w}^2}{\partial T} - e^T \frac{\partial \tilde{w}^1}{\partial X} = 0, \qquad \frac{\partial \tilde{w}^2}{\partial X} - e^T \frac{\partial \tilde{w}^1}{\partial T} = 0,$$

which is the particular linear system obtained from linearization of the NLT system through its admitted infinite set of point symmetries.

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