DIRECT CONSTRUCTION OF CONSERVATION LAWS AND CONNECTIONS BETWEEN SYMMETRIES AND CLs:

Generalizations of Noether's theorem

- Direct method for construction of local CLs
- Relationship between local CL multipliers and solutions of adjoint equations
    - Self-adjoint case
- Noether's theorem
    - Noether's formulation
    - Boyer's generalization
    - Limitations
- Advantages/comments re: direct method vs. Noether's theorem
- Further connections between symmetries and CLs
- References

Uses of Conservation Laws

- Constants of motion; hold for any set of data

- For global convergence, important that CLs are preserved for approximation procedures

- Determine whether a given nonlinear PDE system can be invertibly mapped into a linear PDE system as well as find such a mapping when one exists

- To find equivalent nonlocally related systems of a given PDE system
    - Invariant solutions resulting from point symmetries of nonlocally related system could yield further solutions of given PDE system
    - Computation of CLs of nonlocally related system could yield nonlocal CLs of given PDE system and to non-invertible linearizations, etc

Direct Method for Construction of Conservation Laws

Given system $\mathbf{R}\{x;u\}$ of N PDEs of order k with n indep. var. $x=(x^1,\dots,x^n)$ and m dep. var. $u(x)=(u^1(x),\dots,u^m(x))$:

$$R^\sigma[u]=R(x,t,u,\partial u,\dots,\partial^k u)=0,\quad \sigma=1,\dots,N, \qquad (1)$$

a *local conservation law* (CL) is an expression

$$\mathrm{D}_i\Phi^i[u]=\mathrm{D}_1\Phi^1[u]+\dots+\mathrm{D}_n\Phi^n[u]=0$$

holding for any solutions of (1); D_i, $i=1,\dots,n$, are total deriv. operators

Definition. A PDE system $\mathbf{R}\{x;u\}$ (1) is *totally non-degenerate* if (1) and its differential consequences have maximal rank and are locally solvable.

Theorem. Let $\mathbf{R}\{x;u\}$ (1) be a totally non-degenerate PDE system. Then for every nontrivial local conservation law

$$\mathrm{D}_i\Phi^i[u] = \mathrm{D}_i\Phi^i(x,u,\partial u,\ldots,\partial^r u) = 0$$

of (1), there exists a set of multipliers

$$\Lambda_\sigma[U] = \Lambda_\sigma(x,U,\partial U,\ldots,\partial^l U), \quad \sigma = 1,\ldots,N,$$

such that

$$\mathrm{D}_i\Phi^i[U] \equiv \Lambda_\sigma[U]R^\sigma[U]$$

holds for arbitrary $U(x)$.

Definition. The *Euler operator* with respect to U^j is the operator

$$\mathrm{E}_{U^j} = \frac{\partial}{\partial U^j} - \mathrm{D}_i \frac{\partial}{\partial U_i^j} + \ldots + (-1)^s \mathrm{D}_{i_1} \ldots \mathrm{D}_{i_s} \frac{\partial}{\partial U_{i_1 \ldots i_s}^j} + \ldots$$

Theorem. For **any** divergence expression $\mathrm{D}_i \Phi^i[U]$, one has

$$\mathrm{E}_{U^j}(\mathrm{D}_i \Phi^i[U]) \equiv 0, \quad j = 1, \ldots, m.$$

Theorem. Let $F[U]=F(x,U,\partial U,\ldots,\partial^s U)$. Then

$$\mathrm{E}_{U^j}F[U]\equiv 0,\ j=1,\ldots,m,$$

holds for arbitrary $U(x)$ if and only if

$$F[U]\equiv \mathrm{D}_i\Psi^i(x,U,\partial U,\ldots,\partial^{s-1}U)$$

for some set of functions $\{\Psi^i(x,U,\partial U,\ldots,\partial^{s-1}U)\}$.

Theorem. A set of local multipliers $\{\Lambda_\sigma(x,U,\partial U,\ldots,\partial^l U)\}$ yields a divergence expression for PDE system $\mathbf{R}\{x;u\}$ (1) if and only if

$$\mathrm{E}_{U^j}(\Lambda_\sigma(x,U,\partial U,\ldots,\partial^l U)R^\sigma(x,U,\partial U,\ldots,\partial^k U))\equiv 0,\ j=1,\ldots,m \quad (2)$$

holds for arbitrary $U(x)$.

Summary of direct method to find local CLs

- Seek multipliers of the form

$$\Lambda_\sigma[U] = \Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)$$

with derivatives $\partial^l U$ to some specified order *l*.

- Obtain and solve the determining equations (2) to find the multipliers of local conservation laws

- Find the corresponding fluxes $\Phi^i[U] = \Phi^i(x, U, \partial U, \ldots, \partial^r U)$ satisfying the identity

$$\Lambda_\sigma[U] R^\sigma[U] \equiv \mathrm{D}_i \Phi^i[U], \quad (3)$$

=> CL

$$\mathrm{D}_i \Phi^i[u] = \mathrm{D}_i \Phi^i(x, u, \partial u, \ldots, \partial^r u) = 0$$

with fluxes $\Phi^i[u]$ holding for all solutions of PDE system (1).

The fluxes are found by either

- directly matching each side of

$$\Lambda_\sigma[U]R^\sigma[U] \equiv \mathrm{D}_i\Phi^i[U] \quad (4)$$

[Here $\{\Lambda_\sigma[U]\}$ and $\{R^\sigma[U]\}$ are known with $\{\Phi^i[U]\}$ to be determined.]

- through an integral (homotopy) formula

Example 1-**Nonlinear telegraph system**

$$R_1[u,v] = v_t - (u^2+1)u_x - u = 0,$$
$$R_2[u,v] = u_t - v_x = 0. \qquad (5)$$

Seek CL multipliers of the form

$$\Lambda_1 = \xi[U,V] = \xi(x,t,U,V), \; \Lambda_2 = \phi[U,V] = \phi(x,t,U,V) \quad (6)$$

for (5). In terms of Euler operators

$$\mathrm{E}_U = \frac{\partial}{\partial U} - \mathrm{D}_x \frac{\partial}{\partial U_x} - \mathrm{D}_t \frac{\partial}{\partial U_t}, \quad \mathrm{E}_V = \frac{\partial}{\partial V} - \mathrm{D}_x \frac{\partial}{\partial V_x} - \mathrm{D}_t \frac{\partial}{\partial V_t}$$

the multipliers (6) yield a local CL of (5) if and only if

$$\mathrm{E}_U(\xi[U,V]R_1[U,V]+\phi[U,V]R_2[U,V])=0,$$
$$\mathrm{E}_V(\xi[U,V]R_1[U,V]+\phi[U,V]R_2[U,V])=0, \qquad (7)$$

hold for arbitrary differentiable functions $U(x,t)$, $V(x,t)$. Equations (7) hold if and only if

$$\phi_V-\xi_U=0,$$
$$\phi_U-(U^2+1)\xi_V=0,$$
$$\phi_x-\xi_t-U\xi_V=0, \qquad (8)$$
$$(U^2+1)\xi_x-\phi_t-U\xi_U-\xi=0.$$

The solutions of (8) are:

$$(\xi_1,\phi_1)=(0,1), \quad (\xi_2,\phi_2)=(t,x-\tfrac{1}{2}t^2), \quad (\xi_3,\phi_3)=(1,-t),$$
$$(\xi_4,\phi_4)=(e^{x+\frac{1}{2}U^2+V},Ue^{x+\frac{1}{2}U^2+V}), \quad (\xi_5,\phi_5)=(e^{x+\frac{1}{2}U^2-V},-Ue^{x+\frac{1}{2}U^2-V}).$$

The corresponding five local conservation laws are obtained straightforwardly:

$$\mathrm{D}_t u + \mathrm{D}_x[-v] = 0,$$

$$\mathrm{D}_t[(x - \tfrac{1}{2}t^2)u + tv] + \mathrm{D}_x[(\tfrac{1}{2}t^2 - x)v - t(\tfrac{1}{3}u^3 + u)] = 0,$$

$$\mathrm{D}_t[v - tu] + \mathrm{D}_x[tv - (\tfrac{1}{3}u^3 + u] = 0,$$

$$\mathrm{D}_t[e^{x+\frac{1}{2}u^2+v}] + \mathrm{D}_x[-ue^{x+\frac{1}{2}u^2+v}] = 0,$$

$$\mathrm{D}_t[e^{x+\frac{1}{2}u^2-v}] + \mathrm{D}_x[ue^{x+\frac{1}{2}u^2-v}] = 0.$$

Example 2-**Korteweg-de Vries equation**

$$R[u] = u_t + uu_x + u_{xxx} = 0. \quad (9)$$

It is convenient to also write (9) as $u_t = g[u] = -(uu_x + u_{xxx})$.

Hence all CL multipliers are of the form

$$\Lambda[U] = \Lambda(t, x, U, \partial_x U, ..., \partial_x^l U),\, l = 1,2,....$$

Then

$$\mathrm{E}_U(\Lambda[U](U_t + UU_x + U_{xxx})) \equiv 0$$

if and only if

$$\begin{aligned} &-\mathrm{D}_t\Lambda - U\mathrm{D}_x\Lambda - \mathrm{D}_x^3\Lambda + \\ &(U_t + UU_x + U_{xxx})\Lambda_U - \mathrm{D}_x((U_t + UU_x + U_{xxx})\Lambda_{\partial_x U}) \\ &+ ... + (-1)^l \mathrm{D}_x^l((U_t + UU_x + U_{xxx})\Lambda_{\partial_x^l U} \equiv 0. \quad (10) \end{aligned}$$

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Note that the linear determining equation (10) is of the form

$$\alpha_1 + \alpha_2 U_t + \alpha_3 \partial_x U_t + \ldots + \alpha_{l+2} \partial_x^l U_t \equiv 0 \quad (11)$$

where each α_i depends at most on t, x, U and x-derivatives of U. Since $U(x,t)$ is an arbitrary function, in equation (11), it follows that each of $U_t, \partial_x U_t, \ldots, \partial_x^l U_t$ can be treated as independent variables, and hence $\alpha_i = 0,\ i = 1,\ldots,l+2$

Thus equation (11) splits into an overdetermined linear system of $l + 2$ determining equations for the local multipliers

$$\Lambda(t, x, U, \partial_x U, \ldots, \partial_x^l U),$$

given by

$$\widetilde{\mathrm{D}}_t\Lambda + U\mathrm{D}_x\Lambda + \mathrm{D}_x^3\Lambda = 0, \ (12)$$

$$\sum_{k=1}^{l}(-\mathrm{D}_x)^k\Lambda_{\partial_x^k U} = 0,$$

$$(1-(-1)^q)\Lambda_{\partial_x^q U} + \sum_{k=q+1}^{l}\frac{k!}{q!(k-q)!}(-\mathrm{D}_x)^{k-q}\Lambda_{\partial_x^k U} = 0, \quad q = 1,\ldots,l-1,$$

$$(1-(-1)^l)\Lambda_{\partial_x^l U} = 0,$$

where the “restricted” total derivative operator

$$\widetilde{\mathrm{D}}_t = \tfrac{\partial}{\partial t} + g[U]\tfrac{\partial}{\partial U} + (g[U])_x\tfrac{\partial}{\partial U_x} + \cdots$$

in terms of

$$g[U] = -(UU_x + U_{xxx}).$$

Now suppose $\Lambda = \Lambda(t, x, U)$. Then equations (12) are satisfied and the determining equation (11) becomes

$$(\Lambda_t + U\Lambda_x + \Lambda_{xxx}) + 3\Lambda_{xxU}U_x + 3\Lambda_{xUU}U_x^2 + \Lambda_{UUU}U_x^3 + 3\Lambda_{xU}U_{xx} + 3\Lambda_{UU}U_xU_{xx} = 0. \quad (13)$$

Equation (13) must hold for arbitrary values of x, t, U, U_x, U_{xx}. Hence (13) splits into six equations. Their solution yields the three local multipliers

$$\Lambda_1 = 1, \quad \Lambda_2 = U, \quad \Lambda_3 = tU - x.$$

These yield the divergence expressions

$$U_t + UU_x + U_{xxx} \equiv \mathrm{D}_t U + \mathrm{D}_x(\tfrac{1}{2}U^2 + U_{xx}),$$

$$U(U_t + UU_x + U_{xxx}) \equiv \mathrm{D}_t(\tfrac{1}{2}U^2) + \mathrm{D}_x(\tfrac{1}{3}U^3 + UU_{xx} - \tfrac{1}{2}U_x^2),$$

$$(tU - x)(U_t + UU_x + U_{xxx})$$
$$\equiv \mathrm{D}_t(\tfrac{1}{2}tU^2 - xU) + \mathrm{D}_x(-\tfrac{1}{2}xU^2 + tUU_{xx} - \tfrac{1}{2}tU_x^2 - xU_{xx} + U_x).$$

There is only one additional multiplier of the form

$$\Lambda[U] = \Lambda(x, t, U, U_x, U_{xx})$$

given by

$$\Lambda_4 = U_{xx} + \tfrac{1}{2}U^2.$$

Moreover, one can show that in terms of the recursion operator

$$\mathrm{R}^*[U] = \mathrm{D}_x^2 + \tfrac{1}{3}U + \tfrac{1}{3}\mathrm{D}_x^{-1} \circ U \circ \mathrm{D}_x,$$

the KdV equation has an infinite sequence of local multipliers given by

$$\Lambda_{2n} = (\mathrm{R}^*[U])^n U, \quad n = 1, 2, \ldots,$$

General expression relating local multipliers and solutions of adjoint equations

Consider a system of N PDEs $\mathbf{R}\{x;u\}$ given by

$$R^{\sigma}[u] = R^{\sigma}(x, u, \partial u, \ldots, \partial^{k} u) = 0, \quad \sigma = 1, \ldots, N,$$

with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m)$. Let

$$R^{\sigma}[U] = R^{\sigma}(x, U, \partial U, \ldots, \partial^{k} U), \quad \sigma = 1, \ldots, N,$$

where

$$U(x) = (U^1(x), \ldots, U^m(x))$$

is an *arbitrary* function and $U(x) = u(x)$ solves the system of PDEs $\mathbf{R}\{x;u\}$.

In terms of an *arbitrary* function $V(x) = (V^1(x),\dots,V^m(x)),$ the *linearizing operator* $\mathrm{L}[U]$ associated with the PDE system $\mathbf{R}\{x;u\}$ is given by

$$\mathrm{L}^{\sigma}_{\rho}[U]V^{\rho} \equiv \left[\frac{\partial R^{\sigma}[U]}{\partial U^{\rho}} + \frac{\partial R^{\sigma}[U]}{\partial U^{\rho}_{i}}\mathrm{D}_i + \dots + \frac{\partial R^{\sigma}[U]}{\partial U^{\rho}_{i_1\dots i_k}}\mathrm{D}_{i_1}\dots\mathrm{D}_{i_k} \right] V^{\rho},$$

$$\sigma = 1,\dots,N.$$

and, in terms of an *arbitrary* function $W(x) = (W_1(x),\dots,W_N(x)),$ the *adjoint operator* $\mathrm{L}^*[U]$ associated with the PDE system $\mathbf{R}\{x;u\}$ is given by

$$\mathrm{L}^{*\sigma}_{\rho}[U]W_{\sigma} \equiv \frac{\partial R^{\sigma}[U]}{\partial U^{\rho}}W_{\sigma} - \mathrm{D}_i\left(\frac{\partial R^{\sigma}[U]}{\partial U^{\rho}_{i}}W_{\sigma} \right) + \dots$$

$$+ (-1)^k \mathrm{D}_{i_1}\dots\mathrm{D}_{i_k}\left(\frac{\partial R^{\sigma}[U]}{\partial U^{\rho}_{i_1\dots i_k}}W_{\sigma} \right),$$

$$\rho = 1,\dots,m.$$

In particular, $W_\sigma \mathrm{L}^\sigma_\rho[U]V^\rho - V^\rho \mathrm{L}*^\sigma_\rho[U]W_\sigma$ is a divergence expression.

Let

$$W_\sigma = \Lambda_\sigma[U] = \Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U),\ \sigma = 1,\ldots,N.$$

By direct calculation, in terms of Euler operators:

$$\mathrm{E}_{U^\rho}(\Lambda_\sigma[U]R^\sigma[U]) \equiv \mathrm{L}*^\sigma_\rho[U]\Lambda_\sigma[U] + F_\rho(R[U]) \quad (1)$$

with

$$F_\rho(R[U]) = \frac{\partial \Lambda_\sigma[U]}{\partial U^\rho} R^\sigma[U] - \mathrm{D}_i\left(\frac{\partial \Lambda_\sigma[U]}{\partial U^\rho_i} R^\sigma[U]\right) + \ldots$$

$$\rho = 1,\ldots,m.$$

$$+ (-1)^l \mathrm{D}_{i_1}\ldots\mathrm{D}_{i_l}\left(\frac{\partial \Lambda_\sigma[U]}{\partial U^\rho_{i_1\ldots i_l}} R^\sigma[U]\right),$$

(2)

From (1), it follows that $\{\Lambda^{\sigma}(x, U, \partial U, \ldots, \partial^{l} U)\}_{\sigma=1}^{N}$ is a set of CL multipliers of PDE system $\mathbf{R}\{x;u\}$ if and only if the right hand side of (1) vanishes for arbitrary $U(x)$. Moreover, since (2) vanishes for any solution $U(x) = u(x)$ of $\mathbf{R}\{x;u\}$, it follows that every set of CL multipliers $\{\Lambda^{\sigma}(x, U, \partial U, \ldots, \partial^{l} U)\}_{\sigma=1}^{N}$ of $\mathbf{R}\{x;u\}$ is itself a solution of its adjoint system of PDEs (which is the adjoint of its linearizing system of PDEs) when $U(x) = u(x)$ is a solution of $\mathbf{R}\{x;u\}$, i.e.,

$$\mathrm{L}^{*\sigma}_{\rho}[u]\Lambda_{\sigma}[u] = 0, \quad \rho = 1, \ldots, m.$$

Theorem. Consider a given PDE system $\mathbf{R}\{x;u\}$. A set of local multipliers $\{\Lambda_\sigma(x,U,\partial U,\ldots,\partial^\ell U)\}_{\sigma=1}^N$ yields a local conservation law of $\mathbf{R}\{x;u\}$ if and only if the identity

$$\mathrm{L}^{*}{}_{\rho}^{\sigma}[U]\Lambda_\sigma[U] + \frac{\partial \Lambda_\sigma[U]}{\partial U^\rho} R^\sigma[U] - \mathrm{D}_i\left(\frac{\partial \Lambda_\sigma[U]}{\partial U_i^\rho} R^\sigma[U]\right)$$

$$+\cdots+(-1)^\ell \mathrm{D}_{i_1}\cdots\mathrm{D}_{i_\ell}\left(\frac{\partial \Lambda_\sigma[U]}{\partial U_{i_1\cdots i_\ell}^\rho} R^\sigma[U]\right) \equiv 0, \quad \rho=1,\ldots,m,$$

holds for arbitrary $U(x)=(U^1(x),\ldots,U^m(x))$ in terms of the components $\{\mathrm{L}^{*}{}_{\rho}^{\sigma}[U]\}$ of the adjoint operator of the linearizing operator (Fréchet derivative) for $\mathbf{R}\{x;u\}$.

Theorem. Consider a given PDE system $\mathbf{R}\{x;u\}$. Suppose $\{\Lambda_\sigma(x,U,\partial U,\ldots,\partial^l U)\}_{\sigma=1}^N$ is a set of local multipliers that yields a local conservation law of the PDE system $\mathbf{R}\{x;u\}$. Let $\{\mathrm{L}{*}_\rho^\sigma[U]\}$ be the components of the adjoint operator of the linearizing operator (Fréchet derivative) for the PDE system $\mathbf{R}\{x;u\}$ and let $U(x)=u(x)=(u^1(x),\ldots,u^m(x))$ be any solution of the PDE system $\mathbf{R}\{x;u\}$. Then

$$\mathrm{L}{*}_\rho^\sigma[u]\Lambda_\sigma[u]=0.$$

The situation when the linearizing operator is self-adjoint

Definition. Let $\mathrm{L}[U]$, with its components $\mathrm{L}^{\sigma}_{\rho}[U]$, be the linearizing operator associated with a PDE system $\mathbf{R}\{x;u\}$. The adjoint operator of $\mathrm{L}[U]$ is $\mathrm{L}^*[U]$, with components $\mathrm{L}^{*\sigma}_{\rho}[U]$. $\mathrm{L}[U]$ is a *self-adjoint* operator if and only if $\mathrm{L}[U] \equiv \mathrm{L}^*[U]$, i.e., $\mathrm{L}^{\sigma}_{\rho}[U] \equiv \mathrm{L}^{*\sigma}_{\rho}[U], \sigma, \rho = 1,\ldots,m$

It is straightforward to see that if a PDE system, as written, has a self-adjoint linearizing operator, then

- the number of dependent variables appearing in the system must equal the number of equations appearing in the system, i.e., $N = m$;
- the highest-order partial derivative appearing in the system must be of even order (assuming the adjoint system is not included in the given PDE system)

The converse of the last statement is false. For example, consider the linear heat equation

$$u_{xx} - u_t = 0.$$

Its linearizing operator is $\mathrm{L} = \mathrm{D}_x^2 - \mathrm{D}_t$, with adjoint operator $\mathrm{L}^* = \mathrm{D}_x^2 + \mathrm{D}_t \not\equiv \mathrm{L}$.

One can show that a given PDE system, as written, has a variational formulation if and only if its associated linearizing operator is self-adjoint [Volterra (1913), Vainberg (1964), Olver (1986)].

If the linearizing operator associated with a given PDE system is self-adjoint, then each set of local CL multipliers yields a local symmetry of the given PDE system. In particular, one has the following theorem.

Theorem. Consider a given PDE system **R**$\{x;u\}$ with $N = m$. Suppose its associated linearizing operator L$[U]$ is self-adjoint. Let $\{\Lambda_\sigma(x,U,\partial U,\ldots,\partial^l U)\}_{\sigma=1}^m$ be a set of local CL multipliers for **R**$\{x;u\}$. Let

$$\eta^\sigma(x,u,\partial u,\ldots,\partial^l u) = \Lambda_\sigma(x,u,\partial u,\ldots,\partial^l u),\ \sigma = 1,\ldots,m,$$

where $U = u$ is any solution of the PDE system **R**$\{x;u\}$. Then

$$\eta^\sigma(x,u,\partial u,\ldots,\partial^l u)\frac{\partial}{\partial u^\sigma}$$

is a local symmetry of **R**$\{x;u\}$.

Proof. Since the hypothesis of the previous theorem is satisfied with L[U] = L*[U], from the equations of this theorem, it follows that in terms of the components of the associated linearizing operator L[U], one has

$$\mathrm{L}_{\rho}^{\sigma}[u]\Lambda_{\sigma}(x,u,\partial u,\ldots,\partial^{l}u)=0,\ \rho=1,\ldots,m,\ (2)$$

where $u=\Theta(x)$ is any solution of the given PDE system $\mathbf{R}\{x;u\}$. But the set of equations (2) is the set of determining equations for a local symmetry $\Lambda_{\sigma}(x,u,\partial u,\ldots,\partial^{l}u)\frac{\partial}{\partial u^{\sigma}}$ of PDE system $\mathbf{R}\{x;u\}$. Hence, (1) is a local symmetry of PDE system $\mathbf{R}\{x;u\}$.

The converse of this theorem is false. In particular, suppose $\eta^{\sigma}(x,u,\partial u,\dots,\partial^{l}u)\frac{\partial}{\partial u^{\sigma}}$ is a local symmetry of a PDE system $\mathbf{R}\{x;u\}$ with a self-adjoint linearizing operator L[U]. Let $\Lambda_{\sigma}(x,U,\partial U,\dots,\partial^{l}U)=\eta^{\sigma}(x,U,\partial U,\dots,\partial^{l}U),\ \sigma=1,\dots,m,$ where $U(x)=(U^{1}(x),\dots,U^{m}(x))$ is an arbitrary function. Then it does not necessarily follow that $\{\Lambda_{\sigma}(x,U,\partial U,\dots,\partial^{l}U)\}_{\sigma=1}^{m}$ is a set of local CL multipliers of $\mathbf{R}\{x;u\}$. This can be seen as follows: In the self-adjoint case, the set of local symmetry determining equations is a subset of the set of local multiplier determining equations. Here *each* local symmetry yields a set of local CL multipliers if and only *each* solution of the set of local symmetry determining equations also solves the remaining set of local multiplier determining equations.

To illustrate the situation, consider an example of a nonlinear PDE whose linearizing operator is self-adjoint but the PDE has a point symmetry that does not yield a multiplier for a local CL:

$$u_{tt} - u(uu_x)_x = 0. \quad (1)$$

It is easy to see that PDE (1) has the scaling point symmetry $x \to \alpha x, u \to \alpha u,$ corresponding to the infinitesimal generator

$$\mathrm{X} = (u - xu_x)\frac{\partial}{\partial u}. \quad (2)$$

The self-adjoint linearizing operator associated with PDE (1) is given by

$$\mathrm{L}[U] = \mathrm{D}_t^2 - U^2\mathrm{D}_x^2 - 2UU_x\mathrm{D}_x - 2UU_{xx} - U_x^2.$$

The determining equation for CL multipliers $\Lambda(t,x,U,U_t,U_x)$ is an identity holding for all values of $t,x,U,U_t,U_x,U_{tt},U_{tx},U_{xx},U_{ttt},U_{ttx}$, U_{txx},U_{xxx}, and splits into a system of two equations consisting of

$$\tilde{\mathrm{D}}_t^2\Lambda - U^2\mathrm{D}_x^2\Lambda - 2UU_x\mathrm{D}_x\Lambda - (2UU_{xx}+U_x^2)\Lambda = 0, \quad (3)$$

and

$$2\Lambda_U + \tilde{\mathrm{D}}_t\Lambda_{U_t} - \mathrm{D}_x\Lambda_{U_t} = 0, \quad (4)$$

in terms of the "restricted" total derivative operator

$$\tilde{\mathrm{D}}_t = \tfrac{\partial}{\partial t} + U_t\tfrac{\partial}{\partial U} + U_{tx}\tfrac{\partial}{\partial U_x} + g[U]\tfrac{\partial}{\partial U_t} + U_{txx}\tfrac{\partial}{\partial U_{xx}} + \mathrm{D}_t(g[U])\tfrac{\partial}{\partial U_{tt}}$$

where $g[U] = U(UU_x)_x$.

The first equation (3) is the determining equation for $\Lambda(t, x, u, u_t, u_x)\frac{\partial}{\partial u}$ to be a contact symmetry of the given PDE (1).

If the contact symmetry satisfies the second determining equation then it yields a local multiplier $\Lambda(t, x, U, U_t, U_x)$ of PDE (1).

It is easy to check that the scaling symmetry (2) obviously satisfies the contact symmetry determining equation (3) but does not satisfy the second determining equation (4) when $u(x,t)$ is replaced by an arbitrary function $U(x,t)$.

Hence the scaling symmetry (2) does not yield a local conservation law of PDE (1).

Noether's Theorem

In 1918, Emmy Noether presented her celebrated procedure (*Noether's theorem*) to find local CLs for systems of DEs that admit a variational principle.

When a given DE system admits a variational principle, then the extremals of the associated action functional yield the given DE system (the *Euler-Lagrange equations*). In this case, Noether showed that if a one-parameter local transformation leaves invariant the action functional (action integral), then one obtains the fluxes of a local CL through an explicit formula that involves the infinitesimals of the local transformation and the Lagrangian (Lagrangian density) of the action functional.

Euler-Lagrange equations

Consider a functional $J[U]$ in terms of n independent variables $x=(x^1,\dots,x^n)$ and m arbitrary functions $U=(U^1(x),\dots,U^m(x))$ and their partial derivatives to order k, defined on a domain Ω,

$$J[U]=\int_\Omega L[U]dx=\int_\Omega L(x,U,\partial U,\dots,\partial^k U)dx.$$

The function $L[U]=L(x,U,\partial U,\dots,\partial^k U)$ is called a *Lagrangian* and the functional $J[U]$ is called an *action integral*.

Consider an infinitesimal change of U: $U(x) \to U(x) + \varepsilon v(x)$ where $v(x)$ is any function such that $v(x)$ and its derivatives to order $k-1$ vanish on the boundary $\partial\Omega$ of the domain Ω.

The corresponding change (variation) in the Lagrangian $L[U]$ is given by

$$\delta L = L(x, U + \varepsilon v, \partial U + \varepsilon \partial v, \ldots, \partial^k U + \varepsilon \partial^k v) - L(x, U, \partial U, \ldots, \partial^k U)$$

$$= \varepsilon \left(\frac{\partial L[U]}{\partial U^i} v^i + \frac{\partial L[U]}{\partial U^i_j} v^i_j + \ldots + \frac{\partial L[U]}{\partial U^i_{j_1 \ldots j_k}} v^i_{j_1 \ldots j_k} \right) + O(\varepsilon^2).$$

Let

$$W^l[U,v] = v^i\left(\frac{\partial L[U]}{\partial U_l^i} + \ldots + (-1)^{k-1}\mathrm{D}_{j_1}\ldots\mathrm{D}_{j_{k-1}}\frac{\partial L[U]}{\partial U^i_{lj_1\ldots j_{k-1}}}\right)$$

$$+ v^i_{j_1}\left(\frac{\partial L[U]}{\partial U^i_{j_1\ell}} + \ldots + (-1)^{k-2}\mathrm{D}_{j_2}\ldots\mathrm{D}_{j_{k-1}}\frac{\partial L[U]}{\partial U^i_{j_1 l j_2\ldots j_{k-1}}}\right) + \ldots$$

$$+ v^i_{j_1\ldots j_{k-1}}\frac{\partial L[U]}{\partial U^i_{j_1 j_2\ldots j_{k-1}l}}.$$

After repeatedly using integration by parts, one can show that

$$\delta L = \varepsilon(v^i \mathrm{E}_{U^i}(L[U]) + \mathrm{D}_\ell W^\ell[U,v]) + O(\varepsilon^2),$$

where E_{U^i} is the Euler operator with respect to U^i.

The corresponding variation in the action integral $J[U]$ is given by

$$\delta J = J[U+\varepsilon v] - J[U] = \int_\Omega \delta L dx$$
$$= \varepsilon \int_\Omega (v^i \mathrm{E}_{U^i}(L[U]) + \mathrm{D}_\ell W^\ell[U,v])dx + O(\varepsilon^2)$$
$$= \varepsilon (\int_\Omega v^i \mathrm{E}_{U^i}(L[U])dx + \int_{\partial\Omega} W^\ell[U,v]n^\ell d\sigma) + O(\varepsilon^2)$$

Hence if $U = u(x)$ extremizes the action integral $J[U]$, then the $O(\varepsilon)$ term of δJ must vanish and hence

$$\int_\Omega v^i \mathrm{E}_{u^i}(L[u])dx = 0$$

for an *arbitrary* $v(x)$ defined on the domain Ω.

Hence, if $U = u(x)$ extremizes the action integral $J[U]$ then $u(x)$ must satisfy the PDE system

$$\mathrm{E}_{u^i}(L[u]) = \frac{\partial L[u]}{\partial u^i} + \ldots + (-1)^k \mathrm{D}_{j_1} \ldots \mathrm{D}_{j_k} \frac{\partial L[u]}{\partial u^i_{j_1 \ldots j_k}} = 0, \quad j = 1, \ldots, m. \tag{1}$$

Equations (1) are called the *Euler-Lagrange equations* satisfied by an extremum $U = u(x)$ of the action integral $J[U]$. Thus

Theorem. If a smooth function $U(x) = u(x)$ is an extremum of an action integral $J[U] = \int_\Omega L[U]dx$ with $L[U] = L(x, U, \partial U, \ldots, \partial^k U)$, then $u(x)$ satisfies **the Euler-Lagrange equations** (1).

Noether's formulation of Noether's theorem

Here the action integral $J[U]$ is invariant under the one-parameter Lie group of point transformations

$$(x^*)^i = x^i + \varepsilon\xi^i(x,U) + O(\varepsilon^2), \quad i = 1,\ldots,n,$$
$$(U^*)^\mu = U^\mu + \varepsilon\eta^\mu(x,U) + O(\varepsilon^2), \quad \mu = 1,\ldots,m,$$

with infinitesimal generator $\mathrm{X} = \xi^i(x,U)\dfrac{\partial}{\partial x^i} + \eta^\nu(x,U)\dfrac{\partial}{\partial U^\nu}$,

if and only if

$$\int_{\Omega^*} L[U^*]dx^* = \int_\Omega L[U]dx$$

where Ω^* is the image of Ω under the Lie group of point transformations

$\mathrm{J} = \det(\mathrm{D}_i (x^*)^j) = 1 + \varepsilon \mathrm{D}_i \xi^i (x, U) + O(\varepsilon^2)$ is Jacobian of transf. Then $dx^* = \mathrm{J}dx$. Moreover, $L[U^*] = e^{\varepsilon \mathrm{X}} L[U]$ in terms of the infinitesimal generator X. Consequently, in Noether's formulation, X is a point symmetry of $J[U]$ if and only if

$$0 = \int_\Omega (\mathrm{J} e^{\varepsilon \mathrm{X}} - 1) L[U] dx = \varepsilon \int_\Omega (L[U] \mathrm{D}_i \xi^i (x, U) + \mathrm{X}^{(k)} L[U]) dx + O(\varepsilon^2) \tag{2}$$

holds for arbitrary $U(x)$ where $\mathrm{X}^{(k)}$ is the k-th extended infinitesimal generator. Hence, if X is a point symmetry of $J[U]$, then the $O(\varepsilon)$ term in (2) must vanish. Thus

$$L[U] \mathrm{D}_i \xi^i (x, U) + \mathrm{X}^{(k)} L[U] \equiv 0$$

The one-parameter Lie group of point transformations with inf. gen. X is equivalent to the one –parameter family of transformations

$$(x^*)^i = x^i, \quad i = 1,\ldots,n,$$
$$(U^*)^\mu = U^\mu + \varepsilon[\eta^\mu(x,U) - U_i^\mu \xi^i(x,U)] + O(\varepsilon^2), \quad \mu = 1,\ldots,m, \qquad (3)$$

with k-th extended infinitesimal generator $\hat{X}^{(k)} = \hat{\eta}^\mu[U]\frac{\partial}{\partial U^\mu} + \ldots$

Under transformation (3), $U(x) \to U(x) + \varepsilon v(x)$ has components

$$v^\mu(x) = \hat{\eta}^\mu[U] = \eta^\mu(x,U) - U_i^\mu \xi^i(x,U)$$

Hence $\delta L = \varepsilon \hat{X}^{(k)} L[U] + O(\varepsilon^2)$. Thus

$$\int_\Omega \delta L dx = \varepsilon \int_\Omega \hat{X}^{(k)} L[U] dx + O(\varepsilon^2).$$

Consequently, comparing expressions, after setting

$$v^{\mu}(x) = \hat{\eta}^{\mu}[U] = \eta^{\mu}(x,U) - U_i^{\mu}\xi^i(x,U),$$

it follows that

$$\hat{\mathrm{X}}^{(k)}L[U] \equiv \hat{\eta}^{\mu}[U]\mathrm{E}_{U^{\mu}}(L[U]) + \mathrm{D}_l W^l[U,\hat{\eta}[U]] \quad (*)$$

Lemma. Let $F[U] = F(x,U,\partial U,...,\partial^k U)$ be an arbitrary function of its arguments. Then, in terms of $\mathrm{X}^{(k)}$ and $\hat{\mathrm{X}}^{(k)}$ the following identity holds:

$$\mathrm{X}^{(k)}F[U] + F[U]\mathrm{D}_i\xi^i(x,U) \equiv \hat{\mathrm{X}}^{(k)}F[U] + \mathrm{D}_i(F[U]\xi^i(x,U)).$$

Theorem. Noether's formulation of Noether's Theorem. Suppose a given PDE system $\mathbf{R}\{x;u\}$ is derivable from a variational principle, i.e., the given PDE system is a set of Euler-Lagrange equations whose solutions $u(x)$ are extrema $U(x) = u(x)$ of an action integral $J[U]$ with Lagrangian $L[U]$. Suppose the one-parameter Lie group of point transformations X leaves invariant $J[U]$. Then

(1) $\hat{\eta}^{\mu}[U]\mathrm{E}_{U^{\mu}}(L[U]) = -\mathrm{D}_i(\xi^i(x,U)L[U] + W^i[U,\hat{\eta}[U]])$ (4)

holds for arbitrary functions $U(x)$, i.e., $\{\hat{\eta}^{\mu}[U]\}_{\mu=1}^{m}$ is a set of local CL multipliers of the Euler-Lagrange system;

(2) The conservation law

$$\mathrm{D}_i(\xi^i(x,u)L[u] + W^i[u,\hat{\eta}[u]]) = 0 \quad (5)$$

holds for any solution $u = \Theta(x)$ of the Euler-Lagrange system.

Proof. Let $F[U] = L[U]$ in the identity in the lemma. Then

$$\hat{\mathrm{X}}^{(k)} L[U] + \mathrm{D}_i (L[U] \xi^i (x, U)) \equiv 0 \quad (6)$$

holds for arbitrary functions $U(x)$. Substitution for $\hat{\mathrm{X}}^{(k)} L[U]$ in (6) through (*) yields (4). If $U(x) = u(x)$ solves the Euler-Lagrange system, then the left-hand-side of equation (4) vanishes. This yields the conservation law (5).

Boyer's formulation of Noether's theorem

Boyer (1967) generalized Noether's theorem to find conservation laws arising from invariance under higher-order transformations through a generalization of Noether's definition of invariance of an action integral $J[U]$. Here action integral $J[U]$ is invariant under a one-parameter higher-order transformation if its integrand $L[U]$ is invariant to within a divergence.

Definition. Let $\hat{\mathrm{X}} = \hat{\eta}^{\mu}(x, U, \partial U, \ldots, \partial^{s} U)\dfrac{\partial}{\partial U^{\mu}}$ be the infinitesimal generator of a one-parameter local transformation with extension $\hat{\mathrm{X}}^{\infty}$. Let $\hat{\eta}^{\mu}[U] = \hat{\eta}^{\mu}(x, U, \partial U, \ldots, \partial^{s} U)$. $\hat{\mathrm{X}}$ is a local symmetry of $J[U]$ if and only if

$$\hat{\mathrm{X}}^{\infty} L[U] \equiv \mathrm{D}_i A^i[U] \quad (8)$$

for some set of functions

$$A^i[U] = A^i(x, U, \partial U, \ldots, \partial^{r} U),\; i = 1, \ldots, n.$$

Theorem. Boyer's generalization of Noether's theorem. Suppose a given PDE system $\mathbf{R}\{x;u\}$ is derivable from a variational principle, i.e., the given PDE system is a set of Euler-Lagrange equations whose solutions $u(x)$ are extrema $U(x) = u(x)$ of an action integral $J[U]$ with Lagrangian $L[U]$. Suppose $\hat{\mathrm{X}} = \hat{\eta}^{\mu}[U]\frac{\partial}{\partial U^{\mu}}$ is a local symmetry of $J[U]$.

Then

(1) $\hat{\eta}^{\mu}[U]\mathrm{E}_{U^{\mu}}(L[U]) \equiv \mathrm{D}_i(A^i[U] - W^i[U,\hat{\eta}[U]])$ (9)

holds for arbitrary functions $U(x)$, i.e.,$\{\hat{\eta}^{\mu}[U]\}_{\mu=1}^{m}$ is a set of local CL multipliers of the Euler-Lagrange system;

(2) The conservation law

$$\mathrm{D}_i(W^i[u,\hat{\eta}[u]] - A^i[u]) = 0 \quad (10)$$

holds for any solution $u = \Theta(x)$ of the Euler-Lagrange system

Proof. For the one-parameter local transformation with infinitesimal generator $\hat{\mathrm{X}} = \hat{\eta}^{\mu}[U]\frac{\partial}{U^{\mu}}$, it follows that the corresponding infinitesimal change $U(x) \to U(x) + \varepsilon v(x)$ has components $v^{\mu}(x) = \hat{\eta}^{\mu}[U]$. Consequently,

$$\delta L = \varepsilon \hat{\mathrm{X}}^{\infty} L[U] + O(\varepsilon^2).$$

But

$$\delta L = \varepsilon(\hat{\eta}^{\mu}[U]\mathrm{E}_{U^{\mu}}(L[U]) + \mathrm{D}_i(W^i[U,\hat{\eta}[U]])) + O(\varepsilon^2).$$

Hence it immediately follows that

$$\hat{\mathrm{X}}^{\infty} L[U] \equiv \hat{\eta}^{\mu}[U]\mathrm{E}_{U^{\mu}}(L[U]) + \mathrm{D}_i(W^i[U,\hat{\eta}[U]]) \quad (11)$$

holds for arbitrary functions $U(x)$. Since $\hat{\mathrm{X}} = \hat{\eta}^{\mu}[U]\frac{\partial}{U^{\mu}}$ is a local symmetry of $J[U]$, it follows that equation (8) holds. Substitution for $\hat{\mathrm{X}}^{\infty} L[U]$ in (11) through (8) yields equation (9). If $U(x) = u(x)$ solves the Euler-Lagrange system, then the left-hand-side of equation (9) vanishes. This yields the conservation law (10).

The following theorem shows that any conservation law obtained through Noether's formulation can be obtained through Boyer's formulation.

Theorem. If a conservation law is obtained through Noether's formulation, then the conservation law can be obtained through Boyer's formulation.

Proof. Suppose the one-parameter Lie group of point transformations with inf. gen. X yields a CL. Then equation (6) holds. Consequently,

$$\hat{\mathrm{X}}^{(k)}L[U] = \hat{\mathrm{X}}^{\infty}L[U] = \mathrm{D}_i A^i[U] \quad (12)$$

where $A^i[U] = -\mathrm{D}_i(L[U]\xi^i(x,U))$. But equation (12) is just the condition for X to be a local symmetry of $J[U]$. Consequently, one obtains the same conservation law from Boyer's formulation.

Limitations of Noether's theorem

1 The difficulty of finding variational symmetries. To find variational symmetries of a given DE system arising from a variational principle, first one determines local symmetries $X = \eta^{\sigma}[u]\frac{\partial}{\partial u^{\sigma}}$ of the Euler-Lagrange equations. Then for each such local symmetry, one checks if X leaves invariant the Lagrangian $L[U]$ to within a divergence. [Since all CLs, obtainable by Noether's theorem, arise from multipliers, one can simply use the direct method to find the variational symmetries.]

2 A given system of DEs is not variational as written. A given system of differential equations, as written, is variational if and only if its linearized system (Fréchet derivative) is self-adjoint. Consequently, it is necessary that a given system of DEs, **as written**, must be of even order, have the same number of equations in the system as its number of dependent variables and be non-dissipative to directly admit a variational principle.

3 Artifices may make a given system of DEs variational. Such artifices include:

- The use of multipliers. As an example, the PDE

$$u_{tt} + H'(u_x)u_{xx} + H(u_x) = 0,$$

as written, does not admit a variational principle since its linearized equation $\varsigma_{tt} + H'(u_x)\varsigma_{xx} + (H''(u_x) + H'(u_x))\varsigma_x = 0$ is not self-adjoint. However, the equivalent PDE

$$e^x[u_{tt} + H'(u_x)u_{xx} + H(u_x)] = 0,$$

as written, is self-adjoint!

- The use of a contact transformation of the variables. As an example, the ODE

$$y'' + 2y' + y = 0, \qquad (*)$$

as written, obviously does not admit a variational principle. But the point transformation $x \to X = x, y \to Y = ye^x$, maps ODE (*) into the variational ODE $Y'' = 0$.

It is well-known that every second order ODE, written in solved form, can be mapped into $Y'' = 0$ by some contact transformation but there is no finite algorithm to find such a transformation.

- The use of a differential substitution. As an example, the Korteweg-de Vries (KdV) equation

$$u_{xxx} + uu_x + u_t = 0,$$

as written, obviously does not admit a variational principle since it is of odd order. But the well-known differential substitution

$$u = v_x$$

yields the related transformed KdV equation

$$v_{xxxx} + v_x v_{xx} + v_{xt} = 0$$

which is the Euler-Lagrange equation for an extremum $V = v$ of the action integral with Lagrangian

$$L[V] = \tfrac{1}{2}(V_{xx})^2 - \tfrac{1}{6}(V_x)^3 - \tfrac{1}{2}V_x V_t.$$

4 Noether's theorem is coordinate-dependent. The use of Noether's theorem to obtain a conservation law is coordinate-dependent since the action of a contact transformation can transform a DE having a variational principle to one that does not have one.

On the other hand it is well-known that conservation laws are coordinate-independent in the sense that a contact transformation maps a conservation law into a conservation law.

5 Artifice of a Lagrangian. One should be able to directly find the conservation laws of a given system of DEs directly without the need to find a related action integral whether or not the given system is variational.

Advantages/comments re: the direct method to find CLs

1. **Works for any system of DEs no matter how it is written. Finds all local CLs. [Noether's thm only finds local CLs.]**

2. **The number of dependent variables does not have to equal the number of equations in the system.**

3. **No functional is required unlike for Noether's thm. CLs are constructed directly.**

4. **Multipliers correspond to symmetries if and only if the linearization operator is self-adjoint (N.A.S.C. for action integral to exist, i.e., given system is variational).**

Example: Consider the Klein-Gordon eqn

$$u_{tx} - u^n = 0,\ n \neq 0,1. \quad (1)$$

Eqn (1) has the point symmetry $x^* = \alpha^{1-n} x,\ t^* = t, u^* = \alpha u$

$$\leftrightarrow \mathrm{X} = (u - (1-n) x u_x) \frac{\partial}{\partial u}$$

Eqn (1) is variational with action functional

$$J[U] = \int L[U] dt dx; \quad L[U] = -\tfrac{1}{2} U_t U_x + \tfrac{1}{n+1} U^{n+1}$$

(1) Noether's formulation of Noether's theorem

Let $x^* = \alpha^{1-n} x,\ t^* = t, U^* = \alpha U$. Then

$J[U^*] = J[\alpha U] = \int L[U^*] dt^* dx^* = \alpha^{1-n} \int L[\alpha U] dt dx.$

But $L[\alpha U] = \alpha^{1+n} L[U] \Rightarrow J[U^*] = \alpha^2 J[U] \neq J[U]$

Hence X is not a point symmetry of the action functional $J[U]$ and hence there is no resulting CL from Noether's formulation of N's thm

(2) Boyer's formulation of Noether's theorem

$$\begin{aligned} X^{\infty} L[U] = {} & U^{n}(U - xU_{x}(1-n)) \\ & - \tfrac{1}{2}(U_{x}(U_{t} - xU_{xt}(1-n) + U_{t}(U_{x} - xU_{xx}(1-n))) \quad (2) \end{aligned}$$

The r.h.s. of (2) does not correspond to a divergence. Best way to show this:

$$E_{U}(X^{\infty} L[U]) = 2(U_{xt} + U^{n}) \neq 0.$$

Hence no CL.

(3) Direct method

$E_{U}[(U - xU_{x}(1-n))(U_{tx} - U)] \neq 0$ for an arbitrary function $U(t,x)$

Hence no CL.

Determination of fluxes of local CLs from multipliers

Let $\{\Lambda_\sigma[U] = \Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)\}_{\sigma=1}^N$ be a set of CL multipliers for PDE system $\mathbf{R}\{x;u\}$. Then for arbitrary functions $U(x) = (U^1(x), \ldots, U^m(x))$, one has

$$\Lambda_\sigma[U] R^\sigma[U] = \mathrm{D}_i \Phi^i[U] \quad (*)$$

for some set of functions $\{\Phi^i(x, U, \partial U, \ldots, \partial^r U)\}_{i=1}^n$ to be found.

Two methods:

- Direct method through equating both sides of (*) to find fluxes
- Homotopy operator method

Example of using direct method:

Consider nonlinear wave equation

$$u_{tt} - (c^2(u)u_x)_x = 0 \quad (1)$$

$\Lambda[U] = xt$ is a multiplier of a local CL of (1). Then

$$xt(\mathrm{D}_t(U_t) - \mathrm{D}_x(c^2(U)U_x)) = \mathrm{D}_t(T[U]) - \mathrm{D}_x(X[U]) \quad (2)$$

for some $T[U] = T(x,t,U,U_x,U_t), X[U] = X(x,t,U,U_x,U_t)$

Then (2) becomes

$$\begin{aligned} & xt(U_{tt} - 2c(U)c'(U)(U_x)^2 - c^2(U)U_{xx}) \\ & = (T_t + T_U U_t + T_{U_t} U_{tt} + T_{U_x} U_{tx}) \qquad (3) \\ & \quad + (X_x + X_U U_x + X_{U_t} U_{tx} + X_{U_x} U_{xx}) \end{aligned}$$

Equate to zero the coefficients of $U_{xx}, U_{tt}, U_{tx}, (U_x)^2, U_t, U_x$, rest

This yields straightforwardly

$$T[U] = xtU_t - xU, \quad X[U] = -xtc^2(U)U_x + t\int c^2(U)dU$$

Use of Symmetries to Find New Conservation Laws from Known Conservation Laws

Any symmetry (discrete or continuous) admitted by a given PDE system $\mathbf{R}\{x;u\}$ maps a conservation law of $\mathbf{R}\{x;u\}$ into another conservation law of $\mathbf{R}\{x;u\}$. Usually, the same conservation law of $\mathbf{R}\{x;u\}$ is obtained.

An admitted symmetry of PDE system $\mathbf{R}\{x;u\}$ induces a symmetry that leaves invariant the linear determining system for its multipliers.

Hence, it follows that if we determine the action of a symmetry on a set of multipliers $\{\Lambda_\sigma[U]\}$ for a known conservation law of $\mathbf{R}\{x;u\}$ to obtain another set of multipliers $\{\hat{\Lambda}^\sigma[U]\}$, then a priori we see whether or not a new conservation is obtained for $\mathbf{R}\{x;u\}$.

Suppose the invertible point transformation

$$x = x(\tilde{x}, \tilde{U}), \quad U = U(\tilde{x}, \tilde{U}), \quad (1)$$

with inverse

$$\tilde{x} = \tilde{x}(x, U), \quad \tilde{U} = \tilde{U}(x, U),$$

is a symmetry of PDE system $\mathbf{R}\{x;u\}$. Then for each PDE in $\mathbf{R}\{x;u\}$, one has

$$R^{\alpha}[U] = A^{\alpha}_{\beta}[\tilde{U}]R^{\beta}[\tilde{U}] \quad (2)$$

holding for some $\{A^{\alpha}_{\beta}[\tilde{U}]\}$.

Theorem. Under the point transformation (1), there exist functions $\{\Psi^i[\tilde{U}]\}$ such that

$$\mathrm{J}[\tilde{U}]\mathrm{D}_i\Phi^i[U] = \tilde{\mathrm{D}}_i\Psi^i[\tilde{U}] \quad (3)$$

where the Jacobian determinant

$$\mathrm{J}[\tilde{U}] = \frac{D(x^1,\ldots,x^n)}{D(\tilde{x}^1,\ldots,\tilde{x}^n)} = \begin{vmatrix} \tilde{\mathrm{D}}_1 x^1 & & \cdots & \tilde{\mathrm{D}}_1 x^n \\ & & \cdots & \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{\mathrm{D}}_n x^1 & & \cdots & \tilde{\mathrm{D}}_n x^n \end{vmatrix} \quad (4)$$

and

$$\Psi^{i_1}[\tilde{U}] = \pm \begin{vmatrix} \Phi^1[U] & \Phi^2[U] & \cdots & \Phi^n[U] \\ \tilde{\mathrm{D}}_{i_2} x^1 & & \cdots & \tilde{\mathrm{D}}_{i_2} x^n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{\mathrm{D}}_{i_n} x^1 & & \cdots & \tilde{\mathrm{D}}_{i_n} x^n \end{vmatrix} \quad (5)$$

Theorem. Suppose the point transformation (1) is a symmetry of $\mathbf{R}\{x;u\}$ and $\{\Lambda_\sigma[U]\}$ is a set of multipliers for a CL of $\mathbf{R}\{x;u\}$ with fluxes $\{\Phi^i[U]\}$. Then

$$\hat{\Lambda}_\beta[\tilde{U}]R^\beta[\tilde{U}] = \tilde{\mathrm{D}}_i\Psi^i[\tilde{U}] \quad (7)$$

where

$$\hat{\Lambda}_\beta[\tilde{U}] = \mathrm{J}[\tilde{U}]A^\alpha_\beta[\tilde{U}]\Lambda_\alpha[U], \quad \beta = 1,\ldots,N, \quad (8)$$

with the components of the derivatives in $\{\Lambda_\alpha[U]\}$ expressed in terms of the prolongation of point transformation (1). In (7), $\Psi^i[\tilde{U}]$ is given by determinant (4); in (8): $A^\alpha_\beta[\tilde{U}]$ is obtained from (2), $\mathrm{J}[\tilde{U}]$ is obtained from (3).

After replacing

$\tilde{x}^i$ by x^i, $\tilde{U}^\alpha$ by U^α, etc. in (7), one obtains the following corollary

Corollary. If $\{\Lambda_\alpha[U]\}$ is a set of multipliers yielding a conservation law of PDE system $\mathbf{R}\{x;u\}$ that has the symmetry (1), then $\{\hat{\Lambda}_\beta[U]\}$ yields a set of multipliers for a conservation law of $\mathbf{R}\{x;u\}$ where $\{\hat{\Lambda}_\beta[U]\}$ is given by (8) after replacing $\tilde{x}^i$ by $x^i, \tilde{U}^\sigma$ by $U^\sigma, \tilde{U}_i^\sigma$ by U_i^σ, etc. The set of multipliers $\{\hat{\Lambda}_\beta[U]\}$ yields a new conservation law of PDE system $\mathbf{R}\{x;u\}$ if and only if this set is nontrivial on all solutions $U = u(x)$ of PDE system $\mathbf{R}\{x;u\}$, i.e.

$$\hat{\Lambda}_\beta[u] \not\equiv c\Lambda_\beta[u],\ \beta = 1,\ldots,N,$$ for some constant c.

Now suppose the symmetry (1) is a one-parameter Lie group of point transformations

$$x = x(\tilde{x},\tilde{U};\varepsilon) = e^{\varepsilon\tilde{X}}\tilde{x}, \quad U = U(\tilde{x},\tilde{U};\varepsilon) = e^{\varepsilon\tilde{X}}\tilde{U} \quad (9)$$

in terms of its infinitesimal generator (and extensions)

$$\tilde{\mathrm{X}} = \xi^{j}(\tilde{x},\tilde{U})\tfrac{\partial}{\partial \tilde{x}^{j}} + \eta^{\sigma}(\tilde{x},\tilde{U})\tfrac{\partial}{\partial \tilde{U}^{\sigma}}.$$

If (6) holds, then from (3) and the Lie group properties of (9), it follows that

$$\mathrm{J}[U;\varepsilon]e^{\varepsilon X}(\Lambda_{\sigma}[U]R^{\sigma}[U]) = \mathrm{D}_{i}\Psi^{i}[U;\varepsilon] \quad (10)$$

in terms of the (extended) infinitesimal generator

$$\mathrm{X} = \xi^{j}(x,U)\tfrac{\partial}{\partial x^{j}} + \eta^{\sigma}(x,U)\tfrac{\partial}{\partial U^{\sigma}}.$$

Then, after expanding both sides of Eq. (10) in terms of power series in ε, one obtains an expression of the form

$$\sum \varepsilon^p \hat{\Lambda}_\sigma[U;p] R^\sigma[U] = \sum \varepsilon^p D_i\left(\tfrac{1}{p!}\tfrac{d^p}{d\varepsilon^p}\Psi^i[U;\varepsilon]\right)\Big|_{\varepsilon=0}. \quad (12)$$

Corresponding to the sequence of sets of multipliers

$$\{\hat{\Lambda}_\sigma[U;p]\},\ p = 1,2,\ldots,$$

arising in expression (12), one obtains a sequence of CLs

$$\mathrm{D}_i\left(\tfrac{d^p}{d\varepsilon^p}\Psi^i[u;\varepsilon]\right)\Big|_{\varepsilon=0} = 0, \quad p = 1,2,\ldots$$

for system $\mathbf{R}\{x;u\}$ from its known CL

$$\mathrm{D}_i\Phi^i[u] = 0.$$

EXAMPLE 1

$$v_t + (1 - 2e^{2u})u_x - e^u = 0,$$
$$v_x - u_t = 0$$

has CL multipliers

$$\Lambda_1 = \xi = e^{-\frac{1}{2}(U + t/\sqrt{2})} \sin(\tfrac{1}{2}(V + (x + 2e^U)/\sqrt{2}),$$
$$\Lambda_2 = \phi = -e^{-\frac{1}{2}(U + t/\sqrt{2})}(\sqrt{2}e^U \sin(\tfrac{1}{2}(V + (x + 2e^U)/\sqrt{2}))$$
$$+ \cos(\tfrac{1}{2}(V + (x + 2e^U)/\sqrt{2})))$$

and corresponding fluxes

$$T = -2e^{-\frac{1}{2}(u + t/\sqrt{2})} \cos(\tfrac{1}{2}(v + (x + 2e^u)/\sqrt{2}),$$
$$X = 2e^{-\frac{1}{2}(u + t/\sqrt{2})}(\sqrt{2}e^u \cos(\tfrac{1}{2}(v + (x + 2e^u)/\sqrt{2}))$$
$$- \sin(\tfrac{1}{2}(v + (x + 2e^u)/\sqrt{2})))$$

The given PDE system obviously has the symmetries

$$(t, x, u, v) = (-\tilde{t}, \tilde{x}, \tilde{u}, -\tilde{v}) \quad \text{(reflection)}$$

and

$$(t, x, u, v) = (\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v} + \varepsilon) \quad \text{(translations)}$$

One can show that these symmetries yield three new CLs through

(I) Reflection symmetry applied to above CL
(II) Translation symmetry applied to above CL
(III) Reflection symmetry applied again to CL found in (II)

EXAMPLE 2

$$v_t - (\text{sech}^2 u)u_x + \tanh u = 0,$$
$$v_x - u_t = 0,$$

has CL multipliers

$$\Lambda_1 = \xi = e^x(2x + t^2 - V^2 - 2\log(\cosh U)),$$
$$\Lambda_2 = \phi = 2e^x(V \tanh U - t),$$

and corresponding fluxes

$$T = e^x(2tu - \tfrac{1}{3}v^3 + v(t^2 + 2x - 2\log(\cosh u))),$$
$$X = e^x((v^2 - t^2 - 2x + 2(1 + \log(\cosh u)))\tanh u - 2(vt + u)).$$

This PDE system has the point symmetries

$$\mathrm{X}_1 = \frac{\partial}{\partial t}, \quad \mathrm{X}_2 = v\frac{\partial}{\partial t} + \tanh u\,\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + t\frac{\partial}{\partial v}$$

These symmetries yield three new CLs:

I. The $O(\varepsilon), O(\varepsilon^2)$ terms that result from applying the translation symmetry to the above CL yield two new CLs.

II. The action of the second symmetry X_2 on the new $O(\varepsilon)$ CL yields a third new CL.

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