Local symmetries

Lie’s algorithm for finding point symmetries can be extended to find more general local symmetries admitted by PDEs.

In the extension of Lie’s algorithm, one uses differential consequences of the given PDE system, i.e., invariance of a given PDE system is understood to include its differential consequences.

Here it is important to consider the infinitesimal generators for point symmetries in their ***evolutionary form*** where the independent variables are themselves invariant and the action of a group of point transformations is strictly an action on the dependent variables of the PDE system, so that solutions are ***directly mapped into other solutions*** under the group action.

This allows one to readily extend Lie's algorithm to seek ***contact symmetries*** (only existing for scalar PDEs) where now the components of infinitesimal generators for dependent variables can depend at most on the first derivatives of the dependent variable of a given scalar PDE (if this dependence is at most linear on the first derivatives, then a contact symmetry is a point symmetry).

A contact symmetry is equivalent to a point transformation acting on the space of the given independent variables, the dependent variable and it first derivatives and, through this, can be naturally extended to point transformations acting on the space of the given independent variables, the dependent variable and its derivatives to any finite order greater than one.

Lie's algorithm can be still further extended by allowing the infinitesimal generators in evolutionary form to depend on derivatives of dependent variables to any finite order.

This allows one to calculate symmetries that are called ***higher-order symmetries***.

In the scalar case, contact symmetries are first-order symmetries.

Higher-order symmetries are not equivalent to point transformations acting on a finite-dimensional manifold including the independent variables, the dependent variables and their derivatives to some finite order.

Higher-order symmetries are local symmetries in the sense that the components of the dependent variables in their infinitesimal generators depend at most on a finite number of derivatives of the given PDE system's dependent variables so that their calculation only depends on the local behaviour of solutions of the give PDE system.

Local symmetries include point symmetries, contact symmetries and higher-order symmetries.

Local symmetries are uniquely determined when infinitesimal generators are represented in evolutionary form.

Sophus Lie considered contact symmetries.

Emmy Noether (1918) introduced the notion of higher-order symmetries in her celebrated paper on conservation laws.

The well-known infinite sequences of conservation laws of the Korteweg-de Vries (KdV) and sine-Gordon equations are directly related to admitted infinite sequences of local symmetries obtained through the use of recursion operators [Olver (1977)].

Given a PDE

$$G(x,t,u,\partial u,\ldots,\partial^k u) = 0, \qquad (1)$$

its ***local symmetries of order p,***

$$\eta(x,t,u,\partial u,\ldots,\partial^p u)\,\frac{\partial}{\partial u}$$

$\leftrightarrow$ solutions $\eta(x,t,u,\partial u,\ldots,\partial^p u)$ of its

linearized system (***Fréchet derivative***)

$$\left[\frac{\partial G}{\partial u}\eta + \frac{\partial G}{\partial u_x}\mathrm{D}_x\eta + \frac{\partial G}{\partial u_t}\mathrm{D}_t\eta + \frac{\partial^2 G}{\partial {u_x}^2}(\mathrm{D}_x)^2\eta + \cdots\right]_{\substack{G=0,\\ \mathrm{D}_x G=0,\\ \mathrm{D}_t G=0,\\ \vdots}}$$

$$= 0$$

in terms of ***total derivative operators***

$$\mathrm{D}_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \cdots,$$

$$\mathrm{D}_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + \cdots$$

A ***local symmetry of order p,*** $\eta(x,t,u,\partial u,\ldots,\partial^p u)\frac{\partial}{\partial u}$ (including its natural extension to action on derivatives) maps ***any*** solution $u=\theta(x,t)$ (not an invariant solution) of PDE (1) into a one-parameter (ε) family of solutions

$$u = \left(e^{\varepsilon\left(\eta\frac{\partial}{\partial u}+(\mathrm{D}_x\eta)\frac{\partial}{\partial u_x}+(\mathrm{D}_t\eta)\frac{\partial}{\partial u_t}+\cdots\right)} u \right)\Bigg|_{u=\theta(x,t)}$$

and is equivalent to the transformation

$$x^* = x, \quad t^* = t, \quad u^* = e^{\varepsilon\left(\eta\frac{\partial}{\partial u}+(\mathrm{D}_x\eta)\frac{\partial}{\partial u_x}+(\mathrm{D}_t\eta)\frac{\partial}{\partial u_t}+\cdots\right)} u$$

$$= u + \varepsilon\eta(x, t, u, \partial u, \ldots, \partial^p u) + O(\varepsilon^2)$$

If $p = 1$, then the first order symmetry is equivalent to a ***contact symmetry***

$$x^* = x + \varepsilon \frac{\partial \eta}{\partial u_x} + O(\varepsilon^2),\ t^* = t + \varepsilon \frac{\partial \eta}{\partial u_t} + \cdots,$$

$$u^* = u + \varepsilon\left(u_x \frac{\partial \eta}{\partial u_x} + u_t \frac{\partial \eta}{\partial u_t} - \eta \right) + \ldots,$$

$$u_x{}^* = u_x + \varepsilon\left(-u_x \frac{\partial \eta}{\partial u} - \frac{\partial \eta}{\partial x} \right) + \cdots,$$

$$u_t{}^* = u_t + \varepsilon\left(-u_t \frac{\partial \eta}{\partial u} - \frac{\partial \eta}{\partial t} \right) + \cdots$$

If a first order symmetry has an infinitesimal of the form

$$\eta(x,t,u,\partial u) = \xi(x,t,u)u_x + \tau(x,t,u)u_t - \omega(x,t,u)$$

then it is equivalent to the ***point symmetry***

$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2),\ t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2),$$
$$u^* = u + \varepsilon\omega(x,t,u) + O(\varepsilon^2)$$

Example 1

The heat equation $G = u_{xx} - u_t = 0$

has the point symmetries

$$\mathrm{X}_1 = u_x \frac{\partial}{\partial u}, \quad \mathrm{X}_2 = u_t \frac{\partial}{\partial u}, \quad \mathrm{X}_3 = (xu_x + 2tu_t)\frac{\partial}{\partial u},$$

$$\mathrm{X}_4 = (xtu_x + t^2 u_t + [\tfrac{1}{4}x^2 + \tfrac{1}{2}t]u)\frac{\partial}{\partial u},$$

$$\mathrm{X}_5 = (tu_x + \tfrac{1}{2}xu)\frac{\partial}{\partial u}, \quad \mathrm{X}_6 = u\frac{\partial}{\partial u}$$

Example 2

The KdV equation $G = u_{xxx} + uu_x + u_t = 0$ has an infinite sequence of higher-order symmetries in terms of the recursion operator

$$\mathbf{R} = (\mathrm{D}_x)^2 + \frac{2}{3}u + \frac{1}{3}u_x(\mathrm{D}_x)^{-1}$$

given by

$$(\mathbf{R^n})u_x, \quad n = 0,1,2,...$$

$$\leftrightarrow u_x \frac{\partial}{\partial u}, (uu_x + u_{xxx})\frac{\partial}{\partial u},$$

$$(\tfrac{5}{6}u^2 u_x + 4u_x u_{xx} + \tfrac{5}{3}uu_{xxx} + u_{xxxxx})\frac{\partial}{\partial u}, \ldots$$

Local symmetries can be used to determine

- specific invariant solutions

- a one-parameter family of solutions from “any” known solution

- whether a nonlinear system of DEs can be linearized by an invertible transformation and find the linearization when it exists

- whether an inverse scattering transform exists

- whether a given linear PDE with variable coefficients can be invertibly mapped into a linear PDE with constant coefficients and find such a mapping when it exists