Point symmetries

***Lie's algorithm to find them**

***Lie's classical method to find corresponding invariant solutions**

Lie (1880s): Introduced the notion of invariance of a PDE under continuous groups (one-parameter Lie groups) and used this notion to find special solutions (called ***invariant*, *similarity* or *automodel solutions*)**

- Gave algorithm to find admitted point or contact symmetries from solving related linear systems for infinitesimals **infinitesimal generators** (restricted to act 1:1 on space of independent/dependent variables)

- Point symmetry yields one-parameter family of solutions from a known solution

A one-parameter (ε) Lie group of point transformations $g(\varepsilon)$ acting on a space of two independent variables (x, t) and dependent variable u is of the form

$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2) = e^{\varepsilon \mathrm{X}}x,$$

$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2) = e^{\varepsilon \mathrm{X}}t,$$

$$u^* = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2) = e^{\varepsilon \mathrm{X}}u = U(x,t,u;\varepsilon),$$

in terms of its **infinitesimal generator** $\mathrm{X} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}.$

Group can also be found from its infinitesimal generator by solving the corresponding initial value problem for an autonomous system of first order ODEs:

$$\frac{dx^*}{d\varepsilon} = \xi(x^*, t^*, u^*),$$

$$\frac{dt^*}{d\varepsilon} = \tau(x^*, t^*, u^*),$$

$$\frac{du^*}{d\varepsilon} = \eta(x^*, t^*, u^*),$$

with $u^* = u,\ x^* = x,\ t^* = t$ when $\varepsilon = 0$.

Group naturally extends to action on derivatives

$$(u_x)^* = u_x + \varepsilon\eta^x(x,t,u,u_x,u_t) + O(\varepsilon^2) = e^{\varepsilon \mathrm{X}^{(1)}} u_x \text{etc.}$$

X naturally extends to

$$\mathrm{X}^{(\infty)} = \mathrm{X} + \eta^x \frac{\partial}{\partial u_x} + \eta^t \frac{\partial}{\partial u_t} + \cdots$$

Consequently, one is able to find the set of infinitesimal generators admitted by a given system of PDEs.

As an example, consider the heat equation

$$u_{xx} = u_t .$$

The group $g(\varepsilon)$ leaves invariant the heat equation if and only if

$$[e^{\varepsilon \mathrm{X}^{(\infty)}}(u_{xx} - u_t)]\Big|_{u_{xx}=u_t} = 0 \Leftrightarrow [\eta^{xx} - \eta^t]\Big|_{u_{xx}=u_t} = 0 \Leftrightarrow$$

$$\mathrm{X} = \xi(x,t)\frac{\partial}{\partial x} + \tau(t)\frac{\partial}{\partial t} + [f(x,t)u + g(x,t)]\frac{\partial}{\partial u} \quad \text{with}$$

$$\tau'(t) - 2\xi_x = 0, \quad 2f_x - \xi_{xx} + \xi_t = 0, \quad f_{xx} - f_t = 0,$$
$$g_{xx} - g_t = 0$$

Nontrivial six-parameter group admitted by the heat equation:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t},$$

$$X_4 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - (\tfrac{1}{4}x^2 + \tfrac{1}{2}t)u\frac{\partial}{\partial u},$$

$$X_5 = t\frac{\partial}{\partial x} - \tfrac{1}{2}xu\frac{\partial}{\partial u}, \quad X_6 = u\frac{\partial}{\partial u}$$

Mapping of a solution into one-parameter family of solutions:

Under an admitted group $g(\varepsilon)$, *any solution* $u = \theta(x,t)$ of the heat equation (if not invariant) maps into the one-parameter family of solutions $u = \phi(x,t;\varepsilon)$ satisfying the functional equation

$$u = U(e^{\varepsilon X}x, e^{\varepsilon X}t, \theta(e^{\varepsilon X}x, e^{\varepsilon X}t); -\varepsilon)$$

A **similarity (invariant) solution** $u = \theta(x,t)$ satisfies

$$\phi(x,t;\varepsilon) = \theta(x,t) = U(e^{\varepsilon X}x, e^{\varepsilon X}t, \theta(e^{\varepsilon X}x, e^{\varepsilon X}t); -\varepsilon)$$

which holds if and only if $u = \theta(x,t)$ satisfies the *invariant surface condition*

$$\xi(x,t)u_x + \tau(t)u_t = f(x,t)u,$$

i.e.,

$$\frac{dx}{\xi(x,t)} = \frac{dt}{\tau(t)} = \frac{du}{f(x,t)u}$$

The solution of $\dfrac{dx}{\xi(x,t)} = \dfrac{dt}{\tau(t)}$

yields the similarity variable

$$\zeta(x,t) = \text{const}$$

Finally, one obtains the similarity form

$$u = F(\zeta)G(x,t) \qquad (1)$$

with $G(x,t)$ a specific function of x and t; $F(\zeta)$ an arbitrary function of ζ.

Substitution of (1) into the heat equation is **guaranteed** to yield a reduce ODE satisfied by $F(\zeta)$.

Consider the infinitesimal generator

$$\mathrm{X}_4 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - (\tfrac{1}{4}x^2 + \tfrac{1}{2}t)u\frac{\partial}{\partial u}$$

admitted by the heat equation. Solving the characteristic equations

$$\frac{dx^*}{d\varepsilon} = x^* t^*,$$

$$\frac{dt^*}{d\varepsilon} = (t^*)^2,$$

$$\frac{du^*}{d\varepsilon} = -[\tfrac{1}{4}(x^*)^2 + \tfrac{1}{2}t^*]u^*,$$

with $u^* = u,\ x^* = x,\ t^* = t$ when $\varepsilon = 0,$ yields

$$x^* = \frac{x}{1-\varepsilon t},$$

$$t^* = \frac{t}{1-\varepsilon t}, \qquad (2)$$

$$u^* = \sqrt{1-\varepsilon t}\,\exp\left[-\frac{\varepsilon x^2}{4(1-\varepsilon t)}\right]u.$$

The group of transformations (2) maps any solution (not invariant) $u = \theta(x,t)$ into the one parameter family of solutions

$$u = \phi(x,t;\varepsilon)$$

$$= u^* = \frac{1}{\sqrt{1-\varepsilon t}}\exp\left[\frac{\varepsilon x^2}{4(1-\varepsilon t)}\right]\theta\left(\frac{x}{1-\varepsilon t},\frac{t}{1-\varepsilon t}\right).$$

The invariant solutions $u = \theta(x,t)$ arising from (2) satisfy the invariant surface condition (characteristic PDE)

$$xtu_x + t^2 u_t = -[\tfrac{1}{4}x^2 + \tfrac{1}{2}t]u,$$

i.e.,

$$\frac{dx}{xt} = \frac{dt}{t^2} = \frac{du}{-[\tfrac{1}{4}x^2 + \tfrac{1}{2}t]u}$$

The solution of $\dfrac{dx}{xt} = \dfrac{dt}{t^2}$ yields the similarity variable

$$\zeta(x,t) = \frac{x}{t} = \text{const}$$

Then one obtains the similarity solution form

$$u = \frac{1}{\sqrt{t}} \exp\left(\frac{-x^2}{4t}\right) F(\zeta) \qquad (3)$$

Substitution of (3) into the heat equation yields the reduced ODE

$$F''(\varsigma) = 0$$

$\Rightarrow$ similarity solutions

$$u = \left(C_1 + C_2 \frac{x}{t} \right) \frac{1}{\sqrt{t}} \exp\left(\frac{-x^2}{4t} \right)$$

Buckingham (1914/15): Introduced a systematic procedure to make every real equation *dimensionless*. Consequently, a boundary value problem (BVP) for a PDE might be reduced to a BVP with fewer independent variables.

- Reduced solutions resulting from dimensional analysis will arise as similarity solutions from invariance under scalings of the independent/dependent variables (converse interesting)