Nonlocality in multidimensions

In the multidimensional situation ($n \geq 3$ independent variables), a local conservation law for a given PDE system $\mathbf{R}\{x;u\}$ yields $\frac{1}{2}n(n-1)$ potential variables.

A local symmetry of the resulting potential system ***always*** corresponds to a local symmetry of $\mathbf{R}\{x;u\}$! [This is not the case for n = 2 independent variables.].

To obtain nonlocal symmetries of $\mathbf{R}\{x;u\}$ it is necessary to augment the potential system by a ***gauge constraint***.

Divergence-type CLs and corresponding potential systems

Consider PDE system $\mathbf{R}\{x;u\}$ with N PDEs of order k with $n \geq 3$ independent variables $x = (x^1, \dots, x^n)$ and m dependent variables $u(x) = (u^1(x), \dots, u^m(x))$:

$$R^\sigma[u] = R^\sigma(x, u, \partial u, \dots, \partial^k u) = 0, \quad \sigma = 1, \dots, N. \tag{1}$$

Suppose $\mathbf{R}\{x;u\}$ (1) has a divergence-type CL

$$div\Phi[u] = \mathrm{D}_i \Phi^i[u] \equiv \mathrm{D}_i \Phi^i(x, u, \partial u, \dots, \partial^r u) = 0. \tag{2}$$

From Poincaré's lemma, one has $\frac{1}{2}n(n-1)$ potential variables $v^{jk}(x) = -v^{kj}(x) \Rightarrow$ set of n potential equations

$$\Phi^i[u] \equiv \mathrm{D}_j v^{ij}, \quad i = 1, \ldots, n \tag{3}$$

equivalent to (2).

The corresponding ***potential system*** $\mathbf{S}\{x;u,v\}$ is the union of $\mathbf{R}\{x;u\}$ (1) and the set of potential equations (3).

$\mathbf{S}\{x;u,v\}$ is nonlocally related and equivalent to $\mathbf{R}\{x;u\}$.

Potential system $\mathbf{S}\{x;u,v\}$ has ***gauge freedom***

$$v^{ij} \to \mathrm{D}_k w^{ijk} \tag{4}$$

where $w^{ijk}(x)$ are $\frac{1}{6}n(n-1)(n-2)$ ***arbitrary*** fcns, components of a totally antisymmetric tensor, i.e., $\mathbf{S}\{x;u,v\}$ has an infinite number of point symmetries (***gauge symmetries***)

$$\mathrm{X}_{\text{gauge}} = \mathrm{D}_k w^{ijk}(x)\frac{\partial}{\partial v^{ij}}. \tag{5}$$

As it stands, potential system $\mathbf{S}\{x;u,v\}$ is ***underdetermined*** due to gauge freedom (4).

Now assume that the given PDE system $\mathbf{R}\{x;u\}$ is ***determined*** in the sense that it does not have symmetries that involve *arbitrary functions* of ***all*** independent variables $x=(x^1,\ldots,x^n)$.

Suppose potential system $\mathbf{S}\{x;u,v\}$ has the local symmetry

$$\mathrm{X}=\eta^{\mu}(x,u,\partial u,\ldots,\partial^{P}u,v,\partial v,\ldots\partial^{Q}v)\frac{\partial}{\partial u^{\mu}}+\zeta^{\alpha\beta}[u,v]\frac{\partial}{\partial v^{\alpha\beta}} \qquad (6)$$

Then $\mathbf{S}\{x;u,v\}$ has the local symmetries given by the commutator

$[\mathrm{X}_{\mathrm{gauge}}, \mathrm{X}]$ that projects to the symmetries

$$\left(\alpha^{ij} \frac{\partial \eta^{\mu}}{\partial v^{ij}} + (\mathrm{D}_{i_1}\alpha^{ij}) \frac{\partial \eta^{\mu}}{\partial v^{ij}_{i_1}} + \cdots + (\mathrm{D}_{i_1} \cdots \mathrm{D}_{i_Q}\alpha^{ij}) \frac{\partial \eta^{\mu}}{\partial v^{ij}_{i_1 \cdots i_Q}}\right) \frac{\partial}{\partial u^{\mu}} \qquad (7)$$

of $\mathbf{R}\{x;u\}$ (1) where $\alpha^{ij}(x) = \mathrm{D}_k w^{ijk}(x)$,

and $v^{ij}_{i_1 \cdots i_R} = \mathrm{D}_{i_1} \cdots \mathrm{D}_{i_R} v^{ij}$ denotes derivatives of v^{ij}.

In (7): $\alpha^{ij}(x)$ and each of its derivatives are arbitrary functions of $x = (x^1, \ldots, x^n)$. Since the given PDE system $\mathbf{R}\{x;u\}$ is a ***determined*** system, it follows that (7) is a symmetry of $\mathbf{R}\{x;u\}$ if and only if $\frac{\partial \eta^{\mu}}{\partial v^{ij}} = \frac{\partial \eta^{\mu}}{\partial v^{ij}_{i_1}} = \cdots = \frac{\partial \eta^{\mu}}{\partial v^{ij}_{i_1 \cdots i_Q}} \equiv 0.$

Thus each local symmetry of the ***underdetermined*** potential system **S**{x;u,v} (arising from a divergence-type conservation law) yields only a local symmetry of the given ***determined*** PDE system **R**{x;u}.

Hence if potential system **S**{x;u,v}, arising from a divergence-type conservation law of a given PDE system **R**{x;u}, is used to obtain a potential symmetry of **R**{x;u}, *it is necessary to augment* **S**{x;u,v} with auxiliary constraint equations (***gauge constraints***) to obtain a ***determined potential system***.

A ***gauge constraint*** has the property that the augmented potential system remains equivalent to the given PDE system **R**{x;u}.

Examples of gauges (relating potential variables):

- divergence (Coulomb) gauge
- spatial gauge
- Poincaré gauge
- Lorentz gauge (a form of divergence gauge)
- Cronstrom gauge (a form of Poincaré gauge)

Example

Consider the wave equation **R**{x;u}:

$$u_{tt} - u_{xx} - u_{yy} = 0 \tag{8}$$

which is already a divergence-type CL

Correspondingly, we have vector potential $v = (v^0, v^1, v^2)$ and underdetermined potential system **S**{x;u,v}:

$$\begin{aligned} u_t &= v_x^2 - v_y^1, \\ -u_x &= v_y^0 - v_t^2, \\ -u_y &= v_t^1 - v_x^0 \end{aligned} \tag{9}$$

Now consider the augmented equivalent constrained system obtained by appending the Lorentz gauge

$$v_t^0 - v_x^1 - v_y^2 = 0 \qquad (10)$$

to (9) to obtain the determined potential system

$$\begin{aligned} u_t &= v_x^2 - v_y^1, \\ -u_x &= v_y^0 - v_t^2, \\ -u_y &= v_t^1 - v_x^0, \\ v_t^0 - v_x^1 - v_y^2 &= 0. \end{aligned} \qquad (11)$$

One can show that the determined potential system (11) has six point symmetries that yield nonlocal symmetries as well as nonlocal CLs of the wave equation (8), eg:

$$\mathrm{X} = (yv^1 - xv^2 - tu)\frac{\partial}{\partial u} - (2tv^0 + xv^1 + yv^2)\frac{\partial}{\partial v^0} - (xv^0 + 2tv^1 - yu)\frac{\partial}{\partial v^1} - (yv^0 + 2tv^2 + xu)\frac{\partial}{\partial v^2}$$

One can show that the other listed gauges yield no nonlocal symmetries from point symmetries of the corresponding determined potential systems.

In the multidimensional situation ($n \geq 3$ independent variables), there are three known ways (with known examples) to seek nonlocal symmetries of a given PDE system $\mathbf{R}\{x;u\}$ through seeking local symmetries of an equivalent nonlocally related PDE system:

- Potential systems arising from a divergence-type conservation laws (of degree $r: 1 < r \leq n-1$) augmented with gauge constraints to yield a determined potential system

- Determined potential systems arising from curl-type conservation laws (of degree 1)

- Determined nonlocally related subsystems

In the case of three independent variables (n = 3), two types of CLs arise:

- Degree 2 CLs (divergence-type CL)
- Degree 1 CLs (curl-type CL).

Potential systems arising from lower degree CLs ($r < n-1$) essentially correspond to particular gauge constraints for underdetermined potential systems arising from divergence-type CLs

Examples illustrating the three types of nonlocal symmetries that can arise as described above appear in the following references:

1. Anco and B, Nonlocal symmetries and nonlocal conservation laws of Maxwell’s equations, *J. Math. Phys.* **38** (1997), 3508-3532
2. Anco and The, Symmetries, conservation laws, and cohomology of Maxwell’s equations using potentials, *Acta Appl. Math.* **89** (2005), 1-52.
3. Cheviakov and B, Multidimensional partial differential equation systems: Generating new systems via conservation laws, potentials, gauges, subsystems, *J. Math. Phys.* **51** (2010), 103521.

4. Cheviakov and B, Multidimensional partial differential equation systems: Nonlocal symmetries, nonlocal conservation laws, exact solutions, *J. Math. Phys.* **51** (2010), 103522.
5. Bogoyavlenskij, Infinite symmetries of the ideal MHD equilibrium equations, *Phys. Lett. A*, **291** (2001), 256-264.
6. Bogoyavlenskij, Symmetry transforms for ideal magnetohydrodynamics equilibria, *Phys. Rev. E*, **66** (2002), 056410.
7. B, Cheviakov and Anco, *Applications of Symmetry Methods to Partial Differential Equations* Springer (2010) [Section 5.3]

Some open problems in multidimensions

- Find examples of ***nonlinear*** PDE systems for which nonlocal symmetries arise as local symmetries of a potential system following from divergence-type CLs appended with gauge constraints

- Find efficient procedures to obtain "useful" gauge constraints (eg, yielding nonlocal symmetries/nonlocal CLs) for potential systems arising from divergence-type CLs (as well as for underdetermined potential systems arising from lower-degree CLs). Can one rule out specific families of gauges for particular classes of potential systems?

- Find further examples of lower-degree CLs for PDE systems of physical importance. [CLs of degree one (curl-type) are of particular interest since corresponding potential systems are determined.] Examples to-date suggest that lower-degree CLs are rare and only arise when a given PDE system has a special geometrical structure. Of course, divergence-type CLs are common!

- Find useful subsystems and useful means of obtaining subsystems (including the two-dimensional case). Some progress is being made in this direction.