Construction of Non-Invertible Mappings Relating PDEs

One can use nonlocally related systems to extend work on invertible mappings of given PDEs to ones of simpler type that can draw on an arsenal of well-known solution techniques.

In particular, one can find useful nonlocal mappings relating equivalent PDEs through the use of nonlocally related potential systems.

Firstly, it is shown how to extend the invertible mapping algorithms (based on symmetries or CL multiplers) to nonlocal mappings of nonlinear PDEs to linear PDEs.

Secondly, it is shown how to extend the invertible mapping algorithm to nonlocal mappings of linear PDEs with variable coefficients to linear PDEs with constant coefficients.

Here, one uses the fact that any solution of the adjoint PDE of a given linear PDE yields a multiplier for a conservation law of the given PDE and correspondingly a nonlocally related linear potential system of the given PDE.

The aim is to find such a multiplier that yields an invertible mapping of its corresponding nonlocally related linear system to a constant coefficient linear system.

In turn this yields a non-invertible (nonlocal) mapping of the given linear PDE with variable coefficients to an equivalent linear PDE with constant coefficients.

NON-INVERTIBLE MAPPINGS OF NONLINEAR SYSTEMS OF PDES TO LINEAR SYSTEMS OF PDES

Suppose a given nonlinear system of PDEs does not admit local (point or contact) symmetries (or, equivalently, does not admit multipliers for conservation laws) that yield an invertible mapping to a linear system of PDEs.

However, it could happen that a nonlocally related equivalent nonlinear system of PDEs does admit an infinite set of local symmetries (multipliers for conservation laws) that yield an invertible mapping of the nonlocally related system to some linear system of PDEs.

Consequently, one obtains a nonlocal (non-invertible) mapping of the given nonlinear system of PDEs to an equivalent linear system of PDEs.

Example 1: Linearization of the Thomas Equations

The nonlinear system of Thomas equations given by

$$\begin{aligned} &v_t - u_x = 0, \\ &v_t - uv - u - v = 0, \end{aligned} \qquad (1)$$

describes a fluid flow through a reacting medium. Since (1) does not admit an infinite number of point symmetries, for sure it cannot be linearized by a point transformation. From the first equation of (1), a corresponding potential system is given by

$$\begin{aligned} &w_x = v, \\ &w_t = u, \\ &v_t - uv - u - v = 0. \end{aligned} \qquad (2)$$

One can show that potential system (2) admits the point symmetries

$$\mathrm{X} = e^{w}\left\{[F(x,t)u + H(x,t)]\frac{\partial}{\partial u} + [F(x,t)v + G(x,t)]\frac{\partial}{\partial v} + F(x,t)\frac{\partial}{\partial w}\right\}$$

where $(F(x,t), G(x,t), H(x,t))$ is an arbitrary solution of the linear system of PDEs given by

$$\begin{aligned} F_x &= G, \\ F_t &= H, \\ G_t &= G + H. \end{aligned}$$

Hence one obtains the point transformation

$$z^1 = x,$$

$$z^2 = t,$$

$$w^1 = e^{-w},$$

$$w^2 = e^{-w}v,$$

$$w^3 = e^{-v}u,$$

that 1:1 maps nonlinear system (2) to linear system

$$\frac{\partial w^1}{\partial z^1} = w^2,$$

$$\frac{\partial w^1}{\partial z^2} = w^3,$$

$$\frac{\partial w^2}{\partial z^2} = w^2 + w^3.$$

Consequently, one finds that the solutions $(w^1(z^1,z^2), w^2(z^1,z^2), w^3(z^1,z^2))$ of the linear system yield all solutions, given by

$$(u(x,t), v(x,t)) = -\left(\frac{w^3(x,t)}{w^1(x,t)}, \frac{w^2(x,t)}{w^1(x,t)}\right),$$

of the Thomas equations (1).

From the form of the infinitesimal generator, it follows that the locally related subsystem of (2), given by

$$w_{xt} - w_t w_x - w_t - w_x = 0, \quad (3)$$

admits the linearizing point symmetries

$$\mathrm{X} = F(x,t)e^{w} \frac{\partial}{\partial w}$$

where $F(x,t)$ is any solution of the linear PDE

$$F_{xt} - F_t - F_x = 0.$$

In particular, one obtains the point transformation $W = e^{-w}$ that maps the nonlinear PDE (3) to the linear PDE

$$W_{xt} - W_t - W_x = 0.$$

Example 2: Linearization of a Nonlinear Reaction-Diffusion Equation

Consider the nonlinear reaction-diffusion equation given by

$$u_t - u^2 u_{xx} - 2u^2 = 0. \quad (1)$$

One can show that (1) does not admit linearizing contact symmetries and hence cannot be linearized by an invertible transformation. Multiplying PDE (1) by u^{-2} yields the CL

$$D_t(u^{-1}) + D_x(u_x + 2x) = 0,$$

and corresponding potential system $(u \neq 0)$

$$\begin{aligned} v_x &= u^{-1}, \\ v_t &= -(u_x + 2x) = -(u + x^2)_x. \end{aligned} \quad (2)$$

The nonlinear potential system (2) does not admit linearizing point symmetries. However, since the second PDE in (2) is written as a CL, one accordingly introduces a second potential variable *w* to obtain another nonlocally related potential system

$$\begin{aligned} v_x &= u^{-1}, \\ w_x &= v, \qquad (3) \\ w_t &= -(u + x^2). \end{aligned}$$

One can show that potential system (3) admits linearizing point symmetries

$$\mathrm{X} = e^{(w-xv)}\{(F(t,v) - xH(t,v))\frac{\partial}{\partial x}$$

$$+(G(t,v) - 2xF(t,v) + (x^2 - u)H(t,v))\frac{\partial}{\partial u}$$

$$+(vF(t,v) - (1+xv)H(t,v))\frac{\partial}{\partial w}\},$$

where $(F(t,v), G(t,v), H(t,v))$ is an arbitrary solution of the linear system

$$H_v = F,$$

$$H_t = G,$$

$$F_v = G.$$

Consequently, one can show that the point transformation

$$z^1 = t,$$

$$z^2 = v,$$

$$w^1 = xe^{(xv-w)},$$

$$w^2 = (x^2 + u)e^{(xv-w)},$$

$$w^3 = e^{(xv-w)} - 1,$$

invertibly maps the nonlinear system of PDEs (3) to the linear system

$$\frac{\partial w^1}{\partial z^2} = w^2,$$

$$\frac{\partial w^3}{\partial z^2} = w^1,$$

$$\frac{\partial w^3}{\partial z^1} = w^2.$$

Correspondingly, one can show that a solution $(w^1, w^2, w^3) \neq (0,0,-1)$ of this linear system yields the corresponding solution

$$u = \frac{w^2(w^3+1) - (w^1)^2}{(w^3+1)^2}$$

of the nonlinear reaction-diffusion equation (1).

Example 3--Linearization of a Nonlinear Telegraph Equation

Consider the nonlinear telegraph equation

$$\phi_{tt} = (\phi_t)^2 \phi_{xx} + \phi_t (1 - \phi_t). \qquad (1)$$

One can show that PDE (1) does not admit contact symmetries yielding its linearization by an invertible transformation.

Let $u = \phi_t$, $v = \phi_x$. Then the corresponding PDE system

$$\begin{aligned} u &= \phi_t, \\ v &= \phi_x, \qquad\qquad (2) \\ u_t &= u^2 v_x + u(1-u), \end{aligned}$$

is equivalent to and locally related to the scalar PDE (1), and hence (2) is not linearizable by an invertible transformation.

Clearly, the nonlinear system (2) has a nonlocally related subsystem given by

$$\begin{aligned} u_x &= v_t, \\ u_t &= u^2 v_x + u(1-u). \end{aligned} \qquad (3)$$

As previously shown, the nonlinear telegraph system (3) admits point symmetries yielding its linearization by a point transformation. In turn, this yields the linearization of the nonlinear telegraph equation (1) by a non-invertible (nonlocal) transformation.

Of course, one could consider the nonlinear system of PDEs (3) as the given system with nonlocally related potential system (2) arising from its first equation written as a conservation law. In turn, the scalar equation (1) is a locally related subsystem of the potential system (2).

NON-INVERTIBLE MAPPINGS OF LINEAR PDES WITH VARIABLE COEFFICIENTS TO LINEAR PDES WITH CONSTANT COEFFICIENTS

Previously, we considered the problem of determining whether or not a given linear PDE with variable coefficients can be mapped invertibly to a linear PDE with constant coefficients.

The basis of the presented algorithm was the observation that a linear PDE with constant coefficients is completely characterized by its admitted point symmetries connected with its linearity and invariance under the Abelian group of translations of its independent variables.

This led to a definitive answer to the posed problem and also to the construction of such an invertible mapping when one exists. Parabolic equations were considered as specific examples.

Now suppose a given linear PDE with variable coefficients cannot be mapped invertibly to a linear PDE with constant coefficients.

We now show how to construct non-invertible mappings to extend a class of linear PDEs with variable coefficients that can be mapped to linear PDEs with constant coefficients.

This is accomplished through consideration of an appropriate potential system.

For any given linear PDE system, *any* solution of its adjoint PDE system yields a set of multipliers for a conservation law that yields an equivalent nonlocally related potential system.

The aim is to find such a set of multipliers so that the corresponding potential system can be mapped invertibly into a linear PDE system with constant coefficients..

As a consequence the given linear PDE system is mapped non-invertibly into a constant coefficient linear PDE system.

EXAMPLE: PARABOLIC EQUATION

Let the given PDE be the parabolic PDE in the standard form:

$$\mathrm{L}u = u_{xx} + u_y + V(x,y)u = 0. \qquad (1)$$

PDE (1) can be mapped invertibly by a point transformation to the backward heat equation

$$w_{z^1 z^1} + w_{z^2} = 0$$

if and only if $V(x,y)$ is of the form

$$V(x,y) = a(y)x^2 + b(y)x + c(y)$$

The point transformation that yields the mapping is given by

$$z^1 = \sigma(y)x + \rho(y),$$

$$z^2 = \int^y \sigma^2(\hat{y})d\hat{y},$$

$$w = u\exp\tfrac{1}{4}[\sigma^{-1}\sigma'(y)x^2 + 2\sigma^{-1}\rho'(y)x + \lambda(y)],$$

where $(\sigma(y), \rho(y), \lambda(y))$ is a solution of the nonlinear system of ODEs

$$\begin{aligned} &\sigma^{-2}(\sigma\sigma'' - 2\sigma'^2) = 4a(y), \\ &(\sigma\rho'' - 2\sigma'\rho') = 2\sigma^2 b(y), \\ &\lambda' = \sigma^{-2}(\rho'^2 - 2\sigma\sigma') + c(y). \end{aligned} \tag{2}$$

A multiplier $\phi(x,y)$ yields a CL of PDE (1) if and only if $\phi(x,y)$ is a solution of its adjoint PDE

$$\mathrm{L}^*\phi = \phi_{xx} - \phi_y + V(x,y)\phi = 0. \quad (3)$$

In particular, for arbitrary functions $(U(x,y), \Phi(x,y))$, one has the relationship

$$\begin{aligned}
&\Phi \mathrm{L} U - U \mathrm{L}^* \Phi \\
&\quad = \Phi[U_{xx} + U_y + V(x,y)U] - U[\Phi_{xx} - \Phi_y + V(x,y)\Phi] \\
&\quad = \mathrm{D}_x(\Phi U_x - \Phi_x U) + \mathrm{D}_y(\Phi U).
\end{aligned}$$

Consequently, for **any** solution $\phi(x, y)$ of the adjoint equation (3), the given linear parabolic scalar PDE (1) is nonlocally equivalent to the corresponding linear potential system

$$\begin{aligned} v_x &= \phi u, \\ v_y &= \phi_x u - \phi u_x. \end{aligned} \qquad (4)$$

By direct calculation, one can prove the following extended theorem.

Theorem. Let $\psi(x, y)$ be any solution of the PDE

$$\psi_{xx} + \psi_y + [a(y)x^2 + b(y)x + c(y)]\psi = 0,$$

for some specific coefficients $a(y), b(y), c(y)$. Let $\phi(x, y) = \psi^{-1}$. For the same coefficients $a(y), b(y), c(y),$ consider the parabolic PDE (1) with

$$V(x, y) = -2\frac{\partial^2}{\partial x^2}\log|\phi(x, y)| + a(y)x^2 + b(y)x + c(y), \quad (5)$$

The corresponding potential system (4) can be mapped invertibly by a point transformation to the backward heat potential system

$$\frac{\partial w^2}{\partial z^1} = w^1,$$

$$\frac{\partial w^2}{\partial z^2} = -\frac{\partial w^1}{\partial z^1},$$

Such a mapping is given by

$$
\begin{aligned}
z^1 &= \sigma(y)x + \rho(y), \\
z^2 &= \int^y \sigma^2(\hat{y})d\hat{y}, \\
w^1 &= \sigma^{-1}e^{g(x,y)}\left\{u + \left(\tfrac{1}{2}\sigma^{-1}(\sigma'(y)x + \rho'(y)) - \psi^{-1}\psi_x\right)\psi v\right\}, \\
w^2 &= e^{g(x,y)}\psi v,
\end{aligned}
\tag{6}
$$

where $(\sigma(y), \rho(y), \lambda(y))$ is a solution of the nonlinear system of ODEs (2) and

$$g(x,y) = \tfrac{1}{4}[\sigma^{-1}\sigma'(y)x^2 + 2\sigma^{-1}\rho'(y)x + \lambda(y)].$$

Such a mapping is given by

$$
\begin{aligned}
z^1 &= \sigma(y)x + \rho(y), \\
z^2 &= \int^y \sigma^2(\hat{y})d\hat{y}, \\
w^1 &= \sigma^{-1}e^{g(x,y)}\left\{u + \left(\tfrac{1}{2}\sigma^{-1}(\sigma'(y)x + \rho'(y)) - \psi^{-1}\psi_x\right)\psi v\right\}, \\
w^2 &= e^{g(x,y)}\psi v,
\end{aligned}
\qquad (6)
$$

where $(\sigma(y), \rho(y), \lambda(y))$ is a solution of the nonlinear system of ODEs (2) and

$$g(x,y) = \tfrac{1}{4}[\sigma^{-1}\sigma'(y)x^2 + 2\sigma^{-1}\rho'(y)x + \lambda(y)].$$

The mapping (6) defines a point transformation acting on (x,t,u,v)-space that projects onto a nonlocal transformation acting on (x,t,u)-space if the coefficient of v is nonzero in the third equation of the mapping.

It is easy to see that the mapping (6) yields a nonlocal transformation of PDE (1) with $V(x,y)$ of the form (5), and with $V(x,y)$ not quadratic in x, if and only if $\phi(x,y)$ satisfies the condition

$$\frac{\partial^5}{\partial x^5}\log|\phi(x,y)| \not\equiv 0.$$

Let $\hat{\psi}(z^1, z^2)$ be *any* solution of the backward heat equation $\hat{\psi}_{z^1 z^1} + \hat{\psi}_{z^2} = 0$.

Then from the first set of mapping equations it follows that

$$\psi(x, y) = \hat{\psi}(z_1, z_2)\exp\{-\tfrac{1}{4}[\sigma^{-1}\sigma'(y)x^2 + 2\sigma^{-1}\rho'(y)x + \lambda(y)]\}$$

is a solution of the corresponding parabolic PDE, and accordingly, $V(x, y)$ given by equation (5) becomes

$$V(x,y) = a(y)x^2 + b(y)x + c(y) - 2\sigma^2 \left[\frac{\hat{\psi}_{z_2}}{\hat{\psi}} + \left(\frac{\hat{\psi}_{z_1}}{\hat{\psi}} \right)^2 \right] - \frac{\sigma'(y)}{\sigma(y)}, \quad (7)$$

where $z^1 = \sigma(y)x + \rho(y), z^2 = \int^y \sigma^2(\hat{y})d\hat{y},$ with $\sigma(y), \rho(y)$ related to $a(y), b(y)$ through the first two of ODEs of system (2).

Hence ***every*** solution of the backward heat equation yields a coefficient $V(x,y)$ given by (7) for which the corresponding parabolic PDE (1) can be mapped to the backward heat equation.

Theorem. Let $w=\hat{\psi}(z^1,z^2)$ be a solution of the backward heat equation $w_{z^1z^1}+w_{z^2}=0$. Such a solution yields a coefficient $V(x,y)$ given by (7). The corresponding parabolic PDE (1) can be mapped to the backward heat equation only through a nonlocal transformation if and only if $\hat{\psi}(z^1,z^2)$ is **not** one of the forms

$$\text{(I)}\quad \hat{\psi}(z^1,z^2)=e^{(Pz^1-(P)^2z^2)},$$

$$\text{(II)}\quad \hat{\psi}(z^1,z^2)=\frac{1}{\sqrt{(z^2-\hat{z}^2)}}\exp\left\{\frac{(z^1-\hat{z}^1)^2}{4(z^2-\hat{z}^2)}\right\},$$

where $P,\hat{z}^1,\hat{z}^2$ are arbitrary constants.

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