General Introduction

These Moscow lectures are concerned with some modern developments related to symmetries and conservation laws (CLs) for partial differential equations (PDEs).

They will focus on recent research of B and his collaborators.

Most of the material appears in the 2010 Springer book ***Applications of Symmetry Methods to Partial Differential Equations*** (Bluman/Cheviakov/Anco), Volume 168 Appl. Math. Sciences.

In the latter part of the 19^{th} century, Sophus Lie initiated his studies on continuous groups (Lie groups) in order to put order to, and thereby extend systematically, the hodgepodge of heuristic techniques for solving ODEs.

What Lie showed

*the problem of finding the Lie group of point transformations leaving invariant a DE (***point symmetry*** of a DE), reduces to solving related linear systems of determining equations for the coefficients (*infinitesimals*) of its ***infinitesimal generators***.

*a point symmetry of an ODE leads to *reducing the order* of an ODE (irrespective of any imposed initial conditions)

*a point symmetry of a PDE leads to finding special solutions called ***invariant (similarity) solutions*****.**

*a point symmetry of a DE generates a ***one-parameter family of solutions*** from any known solution of the DE that is not an invariant solution.

Limitations of Lie's work

*restricted number of applications for point symmetries, especially for PDE systems

*few DEs have point symmetries

*the invariant solutions arising from point symmetries yield only a small subset of the solution set of the PDE and hence few posed boundary value problems can be solved.

*difficulty of finding point symmetries

Extensions of Lie's work on PDEs since end of 19th century

*finding further applications of point symmetries to include ***linearizations*** and solutions of boundary value problems

*extending the spaces of symmetries of a given PDE system to include ***local symmetries*** (***higher-order symmetries***) as well as ***nonlocal symmetries***

*extending the applications of symmetries to include ***variational symmetries that*** yield ***conservation laws*** *for variational systems* *extending variational symmetries to ***multipliers*** and resulting conservation laws *for any given PDE system*

*finding further solutions that arise from the extension of Lie’s method to the “***nonclassical method***” as well as other generalizations

*efficiently solving the (overdetermined) linear system of symmetry and/or multiplier determining equations through the development of symbolic computation software as well as related calculations for solving the nonlinear system of determining equations for the nonclassical method.

What is a symmetry of a PDE system and how to find one?

A ***symmetry*** of a PDE system is any transformation of its solution manifold into itself, i.e., *a symmetry transforms (maps) any solution of a PDE system to another solution of the same system.*

Hence continuous symmetries of PDE systems are defined ***topologically*** and not restricted to just point symmetries.

In principle, ***any*** nontrivial PDE has symmetries.

The problem is to ***find and use*** the symmetries of a given PDE system.

Practically, to find symmetries of a given PDE system, one considers transformations, acting ***locally*** on variables of ***a finite-dimensional space***, which leave invariant the solution manifold of the PDE system and its ***differential consequences***.

However, these variables do not have to be restricted to the independent and dependent variables of the given PDE system.

Higher-order symmetries

Higher-order symmetries (***local symmetries***) arise when the solutions of the linear determining equations for infinitesimals depend on a finite number of derivatives of dependent variables of the PDE (infinitesimals for ***point symmetries*** allow at most linear dependence on first derivatives of dependent variables; infinitesimals for ***contact symmetries*** allow arbitrary dependence on at most first derivatives of dependent variables).

In making this extension, it is essential to realize that the ***linear determining equations*** for local symmetries are the ***linearized system*** of the given PDE that holds for ***all*** of its solutions.

Globally, point and contact symmetries act on finite-dimensional spaces whereas higher-order symmetries act on infinite-dimensional spaces consisting of the dependent and independent variables as well as ***all*** of their derivatives.

Well-known integrable equations of mathematical physics such as the Korteweg-de-Vries equation have an ***infinite number of higher-order local symmetries***.

Nonlocal symmetries

Another extension is to consider solutions of the determining equations where infinitesimals have an *ad-hoc dependence on nonlocal variables such as integrals of the dependent variables*.

For some PDEs, such symmetries can be found formally through ***recursion operators*** that depend on inverse differentiation.

Integrable equations such as the sine-Gordon and cubic Schrodinger equations have an ***infinite number of such nonlocal symmetries***.

CLs through variational symmetries (Noether's theorem)

In her celebrated 1918 paper, Emmy Noether showed that if a DE system ***admits a variational principle***, then any local transformation group leaving invariant the action integral for its Lagrangian density, i.e., a ***variational symmetry***, yields a ***local conservation law***.

Conversely, any local CL of a ***variational DE system*** arises from a variational symmetry, and hence there is a direct correspondence between CLs and variational symmetries (Noether's theorem).

Limitations of Noether's theorem

******restricted to variational systems*. A given DE system ***as written***, must be of ***even order***, ***have the same number of dependent variables as the number of equations in the given system***, and have ***no dissipation***.

*In particular, a given DE system, ***as written***, is ***variational*** if and only if its linearized system is ***self-adjoint***.

*difficulty of finding local symmetries of the action integral. In general, not all local symmetries of a variational DE system are variational symmetries.

* use of Noether's theorem to find local conservation laws is coordinate-dependent.

Direct method for finding CLs

A CL of a given DE system is a ***divergence expression*** that vanishes on all solutions of the DE system.

Local CLs arise from scalar products formed by linear combinations of ***local multipliers***, functions of independent and dependent variables and their derivatives with each DE in the system.

This scalar product is annihilated by the ***Euler operators*** associated with each of its dependent variables without restricting these variables in the scalar product to solutions of the system of PDEs, i.e., the dependent variables are now ***arbitrary functions***.

If a given DE system, as written, is variational then local CL multipliers correspond to variational symmetries.

Here local CL multipliers satisfy a system of determining equations that includes the linearizing system of the given DE system ***augmented by additional determining equations*** that taken together correspond to the action integral being invariant under the associated variational symmetry.

More generally, for ***any*** given DE system, all local CL multipliers are the solutions of an easily found linear determining system that includes the ***adjoint system*** of the linearizing DE system.

For any set of local CL multipliers, one can either directly find the fluxes and density of the corresponding local CL and, if this proves difficult, there is an ***integral formula*** that yields them without the need of a specific functional (Lagrangian) even in the case when the given DE system is variational.

Mappings

Another important application of symmetries for PDEs is to determine whether or not a given PDE can be mapped into some equivalent target PDE of interest.

This is especially significant if ***a target class of PDEs can be completely characterized in terms of its symmetries***.

Target classes with such complete characterizations include ***linear PDE systems*** and ***linear PDEs with constant coefficients***.

From knowledge of the point or contact symmetries of a given nonlinear PDE system, *one can determine whether it can be mapped invertibly to a linear PDE system by a point or contact transformation and find such an explicit mapping when one exists*.

Moreover, one can also see whether such a linearization is possible from knowledge of the local ***CL multipliers*** of a given PDE system.

From knowledge of the point symmetries of a linear PDE with variable coefficients, *one can determine whether it can be mapped by an invertible point transformation to a linear PDE with constant coefficients and find such an explicit mapping when one exists*.

Systematic procedure to find nonlocal symmetries through nonlocally related systems

To apply symmetry methods to PDE systems, one needs to work in some specific coordinate frame in order to perform calculations.

A procedure to find symmetries that are ***nonlocal*** and yet are local in some related coordinate frame involves *embedding* a given PDE system in another PDE system obtained by adjoining nonlocal variables in such a way that the related PDE system is equivalent to the given system and the given system arises through projection.

Consequently, any local symmetry of the related system yields a symmetry of the given system (converse also holds).

If the local symmetry of the related system has an essential dependence on the nonlocal variables after projection, then it yields a ***nonlocal symmetry*** of the given PDE system.

A systematic way to find such an embedding is through local CLs of a given PDE system. For each local CL, one can introduce a potential variable(s).

By adjoining the resulting potential equations to the given PDE system, one can construct an *augmented system* (***potential system***) of PDEs.

By construction, such a potential system is nonlocally equivalent to the given PDE system since, through built in *integrability conditions*, any solution of the given PDE system yields a solution of the potential system.

Conversely, through projection any solution of the potential system yields a solution of the given PDE system.

But *this relationship is nonlocal* since there is not a one-to-one correspondence between solutions of the given and potential systems.

If a local symmetry of the potential system has an essential dependence on the potential variables when projected to the given system, then it yields a *nonlocal symmetry* (***potential symmetry***) of the given PDE system.

It turns out that many PDE systems have potential symmetries.

Moreover, one can find other nonlocal symmetries of a given PDE system through seeking local symmetries of an *equivalent subsystem* of the given system or one of its potential systems provided that such a subsystem is nonlocally related to the given PDE system.

Applications of potential systems and nonlocally related subsystems

**Invariant solutions* of potential systems and subsystems can yield further solutions of the given PDE system.

*Since a potential symmetry is a local symmetry of a potential system, it generates a one-parameter family of solutions from any known solution of the potential system that in turn yields a one-parameter family of solutions from a known solution of the given PDE system. Similarly, this is the case for a nonlocal symmetry arising from a subsystem.

*Local CLs of potential systems can yield nonlocal CLs of a given PDE system if their local CL multipliers have an essential dependence on potential variables.

*Linearizations of potential systems through local symmetry or local CL multiplier analysis can yield explicit ***nonlocal linearizations*** of a given PDE system.

*Through a potential system one can extend the mappings of linear systems with variable coefficients to linear systems with constant coefficients to include ***nonlocal mappings*** between such systems.

Further Extensions

*****One can further extend embeddings through using local CLs to systematically construct ***trees of nonlocally related but equivalent systems of PDEs.*** If a given PDE system has n local CLs, then each CL yields potentials and corresponding potential systems.

*From the n local CLs, one can directly construct up to $2^n - 1$ independent nonlocally related systems of PDEs by considering the corresponding potential systems individually (***n singlets***), in pairs (***n*(*n* – 1)/2 *couplets***),…, taken all together (***one n-plet***).

*Any of these $2^n - 1$ systems could lead to the discovery of new nonlocal symmetries and/or nonlocal CLs of the given PDE system or any of the other nonlocally related systems.

*Such nonlocal CLs could yield further nonlocally related systems, etc.

*Subsystems of such nonlocally related systems could yield further nonlocally related systems.

*Correspondingly, ***a tree of nonlocally related systems*** is constructed.

The situation in the case of ***multi-dimensional*** PDE systems is especially interesting.

Here one can show that nonlocal symmetries and nonlocal CLs ***cannot*** arise from potential systems unless they are augmented by ***gauge constraints***.

Applications

Through such constructions, one can *systematically relate* ***Eulerian*** *and* ***Lagrangian*** *coordinate descriptions of gas dynamics and nonlinear elasticity*.

A subsystem of the potential systems arising from system written in Eulerian coordinates (from ***conservation of mass***) yields corresponding systems in Lagrangian coordinates.

For a given class of PDEs with ***constitutive functions***, one finds trees of nonlocally related systems yielding symmetries and CLs with respect to various forms of its constitutive functions.

Comparing the number of local symmetries and the number of CLs of a given PDE system

When a DE system is ***variational***, i.e., its *linearized system is self-adjoint*, then local CLs arise from a subset of its local symmetries and the number of linearly independent local CLs cannot exceed the number of higher-order symmetries.

In general, this *will not be the case* when a system is not variational.

Here a given DE system can have *more local conservation laws than local symmetries as well as vice versa*.

Symmetries mapping CLs to new CLs

For any given PDE system, a transformation group (***continuous*** or ***discrete***) that leaves it invariant yields an explicit formula that maps a CL to a CL of the same system, whether or not the given system is variational.

If the group is continuous, then in terms of a parameter expansion a given CL can map into *more than one* additional CL for the given PDE system.

Invariant solutions and generalizations

A point symmetry of a PDE system maps a solution to a one-parameter family of solutions.

Solutions mapping into themselves are invariant.

Such solutions satisfy the characteristic PDE that is the ***invariant surface condition*** yielding the invariants of the point symmetry.

Invariant solutions arising from point symmetries are the solutions of the given PDE system that satisfy the ***augmented system*** consisting of this characteristic PDE with ***known coefficients*** (obtained from the point symmetry) and the given PDE system itself.

Invariant solutions arise as solutions of a reduced system with one less independent variable.

This method is the ***"classical method"*** of Lie to find invariant solutions of a given PDE system.

“Nonclassical method”

Lie’s “classical method” generalizes to the ***“nonclassical method”*** (B 1967) where one seeks solutions of an augmented system consisting of the given PDE system and the characteristic PDE with ***unknown coefficients*** as well as ***differential consequences of the augmented system.***

The unknown coefficients are determined by substituting the characteristic equation, and its differential consequences, into the determining system for point symmetries of the augmented system.

The resulting over-determined system is ***nonlinear*** (even if the given PDE system is linear) in these unknown coefficients, but *less over-determined* than is the case when finding point symmetries of the given PDE system.

Each solution of the determining system for point symmetries is a solution of the determining system for the unknown coefficients of the characteristic PDE.

Solving for the unknown coefficients, one then proceeds to find the corresponding ***"nonclassical" solutions*** of the augmented system that, by construction, include the classical invariant solutions.

The solutions of a PDE that can be obtained by the nonclassical method include all of its solutions that satisfy a particular functional form (***ansatz***) of some generality that allows an arbitrary dependence on a *similarity variable* (depending on the independent and dependent variables of the PDE) and an arbitrary dependence on a function of a similarity variable and the independent variables of the PDE.

The solutions obtained by the nonclassical method include all solutions obtained “directly” from such an ansatz by the ***direct method*** (Clarkson and Kruskal,1988).

Further extensions related to the classical and nonclassical methods for finding solutions of PDEs include ansatzes corresponding to the ***generalized conditional "symmetry" (GCS)*** method and the method of obtaining ***nonclassical potential solutions***.

These ansatzes are generalizations of the methods of obtaining invariant solutions from higher-order symmetries and potential symmetries in the same way that the nonclassical method generalizes the method of obtaining invariant solutions from point symmetries.

So far both of these ansatzes have not proven to be as effective in obtaining new solutions (especially new explicit solutions) for PDEs as is the case for the ansatz corresponding to the nonclassical method.

Another related generalization to obtain further solutions of PDEs is based on ***group-invariant foliation equations*** where one forms a group-resolving system after converting a given PDE into an equivalent first-order PDE system whose independent and dependent variables, respectively, are given by the classical and differential invariants of an admitted point symmetry of the given PDE.